Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Circuits and Systems Group

EE4C03 STATISTICAL DIGITAL SIGNAL PROCESSING AND MODELING

23 January 2020, 18:30–21:30

Open book exam: copies of the book by Hayes and the course slides allowed.

This exam has four questions (40 points)

Question 1 (10 points)

Assume that $s(n)$ is a stationary random process with zero mean and autocorrelation function $r_s(k) = \delta(k)$. We now form a random process $x(n)$ as follows:

$$x(n) = s(n) + f(n),$$

where $f(n)$ is a known *deterministic* sequence.

Subsequently, we filter the sequence $x(n)$ with $h(n)$, i.e., $y(n) = x(n) * h(n)$, where the system function is given by

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}.$$

(a) Compute the mean and the autocorrelation of $x(n)$.

(b) Is $x(n)$ stationary for every function $f(n)$? If not, can you give a counter example?

Are there functions $f(n)$ for which $x(n)$ is stationary? If so, what are the conditions on $f(n)$ for $x(n)$ to be stationary?

Assume from now on that $f(n) = f$ is a constant.

(c) Compute the power spectrum, $P_x(e^{j\omega})$, of $x(n)$. Make a sketch of this power spectrum.

(d) What is the power spectrum, $P_y(e^{j\omega})$, of $y(n)$? Make a sketch of this power spectrum.

(e) What is the cross-power spectrum, $P_{xy}(e^{j\omega})$, between $x(n)$ and $y(n)$?

Question 2 (12 points)

Suppose we have a random process $x(n)$ with zero mean and *complex-valued* autocorrelation $r_x(k)$ for which we would like to obtain an all-pole model of the form

$$H(z) = \frac{b(0)}{1 + a(1)z^{-1} + a(2)z^{-2}}.$$

(a) Write down the Yule-Walker equations, which express $b(0)$, $a(1)$, and $a(2)$ in terms of $r_x(k)$.

Hint: Don't forget the equation for $b(0)$, and remember that $r_x(k)$ is complex-valued.

(b) For the optimal filter, explain why the denominator of $H(z)$ can be interpreted as a prediction error filter.

(c) What is the orthogonality principle?

Use the orthogonality principle to derive the mean square of the related prediction error, denoted as ϵ.

Suppose now that we have reasons to believe that the signal is periodic and, consequently, the poles of the model should lie on the unit circle. Assume that for our second-order model, we have two poles $e^{j\theta}$ and $e^{-j\theta}$.

(d) Express $a(1)$ and $a(2)$ as a function of θ.

(e) Use the result of (d) and the result of (a) to find relations between θ and $r_x(k)$.

From these relations, derive conditions on $r_x(k)$ such that solving the Yule-Walker equations results in a model with the two mentioned poles.

In a different context, assume now that $r_x(0) = 2$, $r_x(1) = 0.5(1 + j)$, and $r_x(2) = 0.5j$.

(f) Use the Levinson-Durbin recursion to solve the Yule-Walker equations of (a) explicitly.

What are the reflection coefficients?

Is this a stable filter or not (explain why)?

Question 3 (8 points)

Suppose we have a continuous-time random process of the form

$$x(t) = e^{j2\pi f_0 t} + w(t),$$

where $f_0 = 2$ kHz and $w(t)$ is a zero mean process with autocorrelation function $r_w(\tau) = 0.1\delta(\tau)$. Only 5 seconds of the signal have been recorded and are available for processing. We would now like to estimate the power spectrum of $x(t)$ at a resolution of at least 10 Hz.

(a) Suppose we want to estimate the power spectrum at a bandwidth of 2.5 kHz using Bartlett's method of periodogram averaging. What is the minimum sampling rate we require for that, and what is the minimum data segment length (expressed in number of samples) to obtain the desired resolution?

(b) Using the minimum segment length of (a), and 5 seconds of measurements of the signal, how many segments are available for averaging?

(c) How does the sampling rate affect the resolution and variance of your estimate? Are there any benefits to sampling at a higher rate than the one computed in (a)?

(d) Given only the estimated averaged periodogram $\hat{P}_x(e^{j\omega})$, explain how you can estimate the frequency f_0 using MUSIC.

Question 4 (10 points)

Consider a radar system where the distance from the radar to the target is given by x. Assume the target distance x changes with time n, and can thus also be represented by the function $x(n)$. Further, assume the radar system can measure the distance $x(n)$ through a noisy measurement $y(n)$, or in other words, we have

$$y(n) = x(n) + v(n),$$

where $v(n)$ is additive white Gaussian measurement noise with zero mean and variance σ_v^2.

(a) Assume we are at time step n and we have received all measurements from time step 0 up to n, i.e., $y(0), y(1), \ldots, y(n)$. Further, assume we have no knowledge of how the target moves, and hence we simply assume $x(n)$ is a constant x. Then, compute the least squares solution for the target distance x using all the measurements from time step 0 to n. What does this solution represent?

Hint: View x as a filter that filters an input signal that is equal to the constant function 1.

(b) In order to write the solution of the above problem in a recursive way, derive the RLS update equations for $x(n)$. Simplify the equations as much as possible.

What is a good initial condition for the parameters?

(c) Derive the LMS updating rule for $x(n)$.

(d) Is there a step size μ to make this LMS method equivalent to the RLS method? If so, is this step size constant or changing in time?

Hint: First write the RLS update equation in a similar form as the LMS recursion.

(e) Now, assume that we know how the distance of the target changes in time n, and assume we model this as

$$x(n+1) = 2x(n) + w(n),$$

where $w(n)$ is additive white Gaussian process noise with zero mean and variance σ_w^2. Based on this model, derive the Kalman update equations to estimate the target distance $x(n)$.

Will this work better than the other schemes (and why)?