Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Circuits and Systems Group

EE4C03 STATISTICAL DIGITAL SIGNAL PROCESSING AND MODELING

6 November 2019, 13:30–16:30

Open book exam: copies of the book by Hayes and the course slides allowed. No other materials allowed.

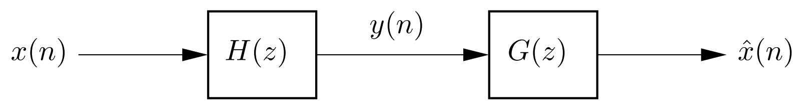
This exam has four questions (40 points)

Question 1 (10 points)

Consider a linear shift-invariant system with input $x(n)$ and output $y(n)$. The input-output relation is given by

$$y(n) = x(n) + \frac{1}{2}x(n-1) + \frac{1}{3}y(n-1).$$

The input $x(n)$ is real-valued zero-mean white noise with variance 1.



(a) Write down the expression for the system transfer function $H(z)$.

(b) What is the power spectral density $P_y(\omega)$ of $y(n)$?

(c) Give an expression for the autocorrelation sequence $r_y(k) = E\{y(n)y(n-k)\}$.

(d) Give an expression for the cross-correlation sequence between the input $x(n)$ and the output $y(n)$, i.e., $r_{xy}(k) = E\{x(n)y(n-k)\}$.

(e) Give expressions for the optimal FIR filter $G(z)$ of order 1 to estimate $x(n)$ from $y(n)$, given knowledge of the correlation functions. (Write this in sufficient detail; you do not have to evaluate the expressions numerically, but make clear how it could be done.)

(f) Does the filter under (e) provide an exact inverse of $H(z)$? How would you determine the best possible inverse filter?

(g) Does the optimal filter $G(z)$ have to change if the input signal $x(n)$ is not white noise but some other random signal?

Question 2 (8 points)

Let $x(n)$ be a random process with autocorrelation sequence $r_x(k) = (0.2)^{|k|}$.

(a) What is the purpose of a prediction error filter?

(b) Find the reflection coefficients Γ_1 and Γ_2 for a second-order predictor related to $x(n)$.

(c) What conclusions can be drawn from the values of these reflection coefficients (e.g. if they are smaller/larger/equal to 1, or if they are zero).

(d) Draw a corresponding lattice filter implementation of the filter.

(e) Draw a filter that, if the input is white noise $v(n)$, generates a signal with autocorrelation sequence $r_x(k)$ (also specify the values of the coefficients).

Question 3 (10 points)

Consider a real-valued signal $x(n)$, which can be modeled as

$$x(n) = c\,d(n) + v(n),$$

where c is a real-valued deterministic parameter and $d(n)$ and $v(n)$ are real-valued mutually uncorrelated random processes with zero mean and respective variances σ_d^2 and σ_v^2.

We attempt to reconstruct $d(n)$ from $x(n)$ using the simple (zero-order) filtering operation

$$\hat{d}(n) = w\,x(n)$$

where we will adapt the coefficient w until it is optimal.

(a) Under which circumstances is a zero-order filter appropriate?

(b) Give the expression of the optimal Wiener filter w_{opt} and explain what happens if σ_v^2 is very small or very large.

(c) Give the update equation for the RLS filter, and try to write the new estimate of the filter $w(n+1)$ as a function of the previous estimate $w(n)$. Clearly define all the notations that you use.

(d) Assume now the following state space model

$$\begin{aligned} w(n+1) &= w(n), \\ d(n+1) &= x(n+1)w(n+1) + e(n+1), \end{aligned}$$

where we assume that $e(n)$ is a random process with zero mean and variance $\sigma_e^2 = 1$. Derive the Kalman filter for this state space model and again clearly define all used notation. Show that this filter is equivalent to the RLS filter derived earlier.

(e) How should the above state space model be adapted to make it equivalent to the exponentially weighted RLS filter?

Question 4 (12 points)

Consider a signal $z(n)$ consisting of two complex sinusoids in noise:

$$z(n) = A_1 e^{j(\omega_1 n + \phi_1)} + A_2 e^{j(\omega_2 n + \phi_2)} + w(n)$$

where ω_1, ω_2 are the frequencies of the sinusoids, A_1, A_2 are their amplitudes, ϕ_1, ϕ_2 are random phases, considered uniformly distributed in $(0, 2\pi)$, and $w(n)$ is zero mean white noise. We have N samples $n = 0, \cdots, N-1$ and create a periodogram.

(a) What is the power spectrum $P_z(e^{j\omega})$ and the expected value of the periodogram $\hat{P}_z(e^{j\omega})$?

(b) Make a sketch of the periodogram.

(c) How can we estimate the frequencies ω_1, ω_2? How does the finite size of N limit the resolution? What is the effect of the noise? What happens if N becomes very large?

Suppose now that a digital communications signal $x(n)$ is transmitted to a receiver over a channel with two reflection paths. The received sampled signal $y(n)$ consists of two attenuated and delayed copies of $x(n)$, and additive white noise:

$$y(n) = \alpha_1 x(n - n_1) + \alpha_2 x(n - n_2) + w(n),$$

where $\alpha_1 = A_1 e^{j\phi_1}$ and $\alpha_2 = A_2 e^{j\phi_2}$. Here, A_1, A_2 are attenuations, ϕ_1, ϕ_2 are random phase offsets, n_1 and n_2 are the delays of the reflection paths, and $w(n)$ is zero mean complex-valued white noise. The input signal $x(n)$ is a random BPSK signal (i.e., $x(n) \in \{-1, +1\}$).

We would like to estimate the delay of the propagation paths from knowledge of $x(n)$ and $y(n)$.

(d) Let $X(e^{j\omega})$ be the DTFT of $x(n)$, and likewise for $Y(e^{j\omega})$ and $W(e^{j\omega})$.

From the data, we compute

$$Z(e^{j\omega}) := \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

What is a model for $Z(e^{j\omega})$ in terms of $X(e^{j\omega})$ and $W(e^{j\omega})$?

Hint: Hayes p. 14 has a table of DTFT properties.

(e) If we only have available N samples, we replace the DTFT by the DFT and obtain the sequence $X[k] = X(e^{j\frac{2\pi}{N}k})$ for $k = 0, \cdots, N-1$, and likewise for $Y[k]$ and $W[k]$. From the data, we compute

$$Z[k] := \frac{Y[k]}{X[k]}$$

- Show that an (approximate) model for $Z[k]$ is

$$Z[k] = \alpha_1 e^{-jk\omega_1} + \alpha_2 e^{-jk\omega_2} + E[k]$$

with $\omega_1 = 2\pi n_1/N$ and $\omega_2 = 2\pi n_2/N$, and $E[k] = \frac{W[k]}{X[k]}$.

- What limits the accuracy of this model?

(f) Is it reasonable to model $W[k]$ as zero mean white noise? Is it reasonable to model $E[k]$ as zero mean white noise? (Motivate your answers.)

(g) Using periodograms, how can we estimate the delays n_1 and n_2 from $Z[k]$, $k = 0, \cdots, N-1$? *Hint: compare to your answers in part (a)-(c).*

How does the finite size of N limit the resolution? What is the effect of the noise? What happens if N becomes very large?