Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

EE4C03 STATISTICAL DIGITAL SIGNAL PROCESSING AND MODELING

24 January 2019, 18:30–21:30

Open book exam: copies of the book by Hayes and the course slides allowed. No other materials allowed.

This exam has four questions (40 points)

Question 1 (8 points)

Given the power spectral density: $P_x(e^{j\omega}) = 5 + 4\cos(\omega)$

(a) Find the corresponding autocorrelation sequence $r_x(k)$.

(b) Find a linear filter whose output process has this autocorrelation sequence, when excited by white noise.

(c) Is this filter a minimum-phase filter? Is this necessarily so?

(d) What is the variance of the output process?

Question 2 (10 points)

Let $x(n)$ be a zero-mean real-valued wide-sense stationary random process with autocorrelation sequence $r_x(k) = 0.5^{|k|}$. This process is modeled by an all-pole filter, excited by white noise.

(a) Write down the expression for an all-pole filter of order 2 for $x(n)$, and the related Yule-Walker equations for finding the coefficients of the model.

(b) Solve the Yule-Walker equations using the Levinson recursion. For every iteration step, specify the corresponding modeling error $E\{(x(n) - \hat{x}(n))^2\}$ and reflection coefficients. Give the final model parameters and modeling error for the all-pole model of order 2.

(c) Is the resulting model stable? Why?

(d) What is the smallest order of an all-pole model for $x(n)$, above which the modeling error does not improve? Is this minimum modeling error equal to zero? If so, why? And if not, why is it not possible to reduce the modeling error to zero by increasing the model order?

Question 3 (11 points)

Assume that we measure a real-valued signal $x(n)$ as $\bar{x}(n)$, and assume that the measurement process is perfect except for some *outliers*, i.e., $\bar{x}(n)$ is equal to $x(n)$ except for a few isolated values of n where $\bar{x}(n)$ is completely different from $x(n)$. Instead of eliminating those values, suppose that we perform a minimum mean-square interpolation as follows. If an outlier appears at time instant $n = n_0$, i.e., $|\bar{x}(n_0) - x(n_0)|$ is very large, we estimate $x(n_0)$ as

$$\hat{x}(n_0) = a\bar{x}(n_0 - 1) + b\bar{x}(n_0 + 1) = ax(n_0 - 1) + bx(n_0 + 1),$$

where the last equality is due to the fact that every outlier is isolated and thus we have no outliers at time instants $n_0 - 1$ and $n_0 + 1$.

(a) Assume that $x(n)$ is a zero-mean real-valued wide-sense stationary random process with autocorrelation sequence $r_x(k)$. Derive the Wiener-Hopf equations for the values of a and b that minimize the mean-square error

$$\xi = E\{|x(n_0) - \hat{x}(n_0)|^2\}.$$

Also derive an expression for the minimum mean-square error. Note that you have to give a *derivation* here and not simply copy equations from the book.

Hint: you will need the following matrix identity: $\begin{bmatrix} p & q \\ q & p \end{bmatrix}^{-1} = \frac{1}{p^2-q^2}\begin{bmatrix} p & -q \\ -q & p \end{bmatrix}$

(b) If $r_x(k) = 0.5^{|k|}$, evaluate the expressions for the optimal a and b, as well as for the minimum mean-square error.

(c) Discuss when it may be better to use an estimator of the form

$$\hat{x}(n_0) = a\bar{x}(n_0 - 1) + b\bar{x}(n_0 - 2) = ax(n_0 - 1) + bx(n_0 - 2),$$

or explain why such an estimator will never be better.

(d) Suppose you want to develop a method to compute a and b adaptively. How would you design such a method? What would you use as input sequence for the adaptive method?

Question 4 (11 points)

Let $x(n)$ be a random process consisting of a single complex exponential in white noise, with covariance sequence

$$r_x(k) = Pe^{jk\omega_0} + \sigma_w^2\delta(k)$$

and $(p+1) \times (p+1)$ autocorrelation matrix $\mathbf{R}_x$.

In the context of minimum-variance spectrum estimation, let $\mathbf{g}_i$ be a vector with the FIR filter coefficients of the minimum-variance bandpass filter,

$$\mathbf{g}_i = \frac{\mathbf{R}_x^{-1}\mathbf{e}_i}{\mathbf{e}_i^H\mathbf{R}_x^{-1}\mathbf{e}_i}, \qquad \mathbf{e}_i = [1, e^{j\omega_i}, e^{j2\omega_i}, \cdots, e^{jp\omega_i}]^T.$$

Denote by $G_i(e^{j\omega}) = \mathbf{e}^H\mathbf{g}_i$, with $\mathbf{e} = [1, e^{j\omega}, e^{j2\omega}, \cdots, e^{jp\omega}]^T$, the corresponding filter response.

(a) Explain how these bandpass filters are used in the estimation of the power spectrum of $x(n)$.

(b) Motivate that $G_i(e^{j\omega})$ is a filter that has center frequency ω_i with $G_i(e^{j\omega_i}) = 1$.

Hint: first determine and plot the filter response for a simple case where $\omega_i = 0$ *and* $\mathbf{R}_x = \mathbf{I}$. *Then motivate what changes in more general cases.*

(c) Using the matrix inversion lemma, we can derive that

$$\mathbf{R}_x^{-1} = \frac{1}{\sigma_w^2}\left(\mathbf{I} - \frac{\mathbf{e}_0\mathbf{e}_0^H P}{\sigma_w^2 + \mathbf{e}_0^H\mathbf{e}_0 P}\right).$$

Use this to determine a (very simple) expression for the filter response for $\omega_i = \omega_0$; give a plot of this response.

(d) Assuming that $\omega_i \neq \omega_0$, prove that $G_i(z)$ has a zero that approaches $z = e^{j\omega_0}$ as $\sigma_w^2 \to 0$.

(e) Give an explanation/interpretation for this.