Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Circuits and Systems Group

EE4C03 STATISTICAL DIGITAL SIGNAL PROCESSING AND MODELING

9 November 2018, 13:30–16:30

Open book exam: copies of the book by Hayes and the course slides allowed. No other materials allowed.

This exam has four questions (40 points)

Question 1 (10 points)

A discrete-time random process $x(n)$ is generated by

$$x(n) = \sum_{k=1}^{p} a(k)x(n-k) \; + \; w(n)$$

where $w(n)$ is a zero mean white noise process with variance σ_w^2. Another process, $y(n)$, is formed by adding noise to $x(n)$,

$$y(n) = x(n) + v(n)$$

where $v(n)$ is zero mean white noise with a variance of σ_v^2 that is uncorrelated with $w(n)$.

(a) What is the power spectrum $P_x(e^{j\omega})$ of $x(n)$.

(*Hint:* first derive z-transform expressions.)

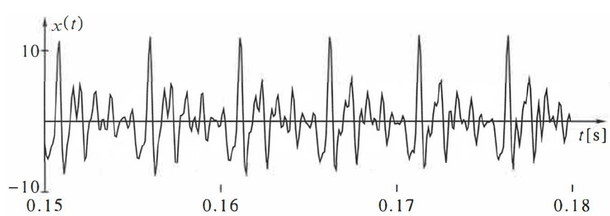
(b) What is the power spectrum $P_y(e^{j\omega})$ of $y(n)$.

(c) What are the Yule-Walker equations for the autocorrelation sequence $r_x(k)$?

Write them in a matrix form showing clearly how $r_x(k)$ $(k = 0, \cdots, p)$ can be computed.

(d) Consider $p = 1$. Derive a closed-form expression for the autocorrelation sequence $r_x(k)$ of $x(n)$.

Question 2 (10 points)



The above signal is a 30 ms segment of voiced speech. Clearly it is quasi-periodic. The time between two pulses is the *pitch period* n_0. For the moment, we assume that n_0 is known.

We aim to model this signal as the output of an all-pole filter $H(z)$ (the vocal tract), driven by a pulse train of delta spikes (the vocal chords).

Let $x(n)$ be a subsequence of the signal consisting of two periods, length $N = 2n_0$, define the input sequence

$$v(n) = \delta(n) + \delta(n - n_0)$$

(two pulses spaced by n_0), and define the AR filter model

$$H(z) = \frac{b(0)}{1 + \sum_{k=1}^{p} a(k)z^{-k}} .$$

The signal model is $\hat{x}(n) = v(n) * h(n)$.

The Prony error is given by $\mathcal{E} = \sum_{n=0}^{N-1} e^2(n)$, where

$$e(n) = a(n) * x(n) - b(n) * v(n)$$

and $b(n) = b(0)\delta(n)$.

(a) Give a clear motivation for this choice of the Prony error.

(b) Derive the Prony normal equations that follow by minimizing the Prony error, resulting in conditions for $a(k)$ $(k = 1, \cdots, p)$ and $b(0)$.

(c) Give explicit (matrix) expressions from which we can directly compute $a(k)$ $(k = 1, \cdots, p)$ and $b(0)$ from the data.

(d) Now suppose the pitch period n_0 is not known. What is a simple and effective method to estimate n_0 from the data? (Note: deriving n_0 from the distance between the peaks of $x(n)$ is not always sufficiently accurate — what is a better method?)

Question 3 (7 points)

Let $y(t)$ be a signal of the form

$$y(t) = \sin(2\pi f_1 t + \phi_1) \; + \; \sin(2\pi f_2 t + \phi_2) \; + \; w(t), \quad t \in [0, T]$$

where $w(t)$ is a zero mean white Gaussian noise process of unit variance and ϕ_1, ϕ_2 are unknown phase components. Also the frequencies f_1 and f_2 are unknown.

You are provided with T seconds of this signal and the task is to estimate its spectrum using the Bartlett method. The parameter that you can control is the sampling frequency, f_s.

(a) We want to attain the following objectives:

- The two frequencies f_1 and f_2 are sufficiently resolved.
- The variance of the spectral estimate is less than 1% of the square of the true spectral density.

Derive an expression for the lowest possible f_s that attains these objectives.

Now, assume that the noise variance σ_w^2 is unknown and that the sampling frequency f_s is given. Assume also that you have the option to choose between using the Periodogram method and using the Bartlett method.

(b) Which of these two alternatives do you recommend if you are primarily interested in estimating the noise level σ_w^2. Motivate your answer.

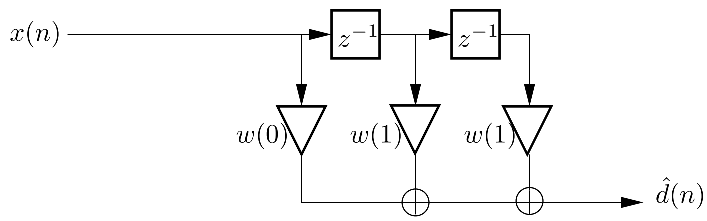
(c) Which of these two alternatives do you recommend if you are primarily interested in getting good estimates of f_1 and f_2, and that $f_2 - f_1$ may be very small. Motivate your answer.

Question 4 (13 points)

Let $x(n)$ and $d(n)$ be two zero-mean real-valued random processes, with auto- and crosscorrelation coefficients given in the following table:

k	0	1	2	3	4	...
$r_d(k) = E\{d(n)d(n-k)\}$	1	0	0	0	0	...
$r_x(k) = E\{x(n)x(n-k)\}$	2	1	1	1	1	...
$r_{dx}(k) = E\{d(n)x(n-k)\}$	1	0	0	0	0	...

Assuming we have measured $x(n)$, we wish to estimate the unknown process $d(n)$ from $x(n)$ using the following FIR filtering structure:



The filter has three tap coefficients, but $w(1)$ and $w(2)$ are equal. The goal is to find the two filter coefficients $w(0)$ and $w(1)$ that minimize the mean square error (MSE)

$$\xi = E\{|d(n) - \hat{d}(n)|^2\}$$

(a) In view of the correlation table, mention a specific application of the resulting filter.

(b) Derive the Wiener-Hopf equations and their solution to determine the optimal filter coefficients $w(0)$ and $w(1)$, as well as the expression for the minimum MSE ξ.

(c) Assuming the values of $r_d(k)$, $r_x(k)$, and $r_{dx}(k)$ as given in the table, what are the values of these optimal filter coefficients and for the minimum MSE?

(d) In the above, we constrained the structure of the filter to satisfy $w(1) = w(2)$. Why does this make sense?

(Hint: check the equations for the unstructured case and compare.)

(e) Suppose we want to compute $w(0)$ and $w(1)$ in an adaptive fashion. Assume for the moment that we have a reference signal $d(n)$ available. Give the expression for the least mean squares (LMS) update equations for $w(0)$ and $w(1)$. Also give appropriate initial conditions and derive a good range for the step size.

(f) Similarly, give the expressions for the exponentialy weighted recursive least squares (RLS) update equations for $w(0)$ and $w(1)$. Also give appropriate initial conditions and a good range for the weighting factor.

(g) In practice, we do not have $d(n)$ available as a reference signal. How can we implement the adaptive algorithms in this case?