Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Circuits and Systems Group

EE4C03 STATISTICAL DIGITAL SIGNAL PROCESSING AND MODELING

25 January 2018, 18:30-21:30

Open book exam: copies of the book by Hayes and the course slides allowed. No other materials allowed.

This exam has four questions (40 points)

Question 1 (10 points)

Given the random signal $x[n]$:

$$x[n] = Ae^{j\omega_0 n+\phi} + w[n]$$

where A is a deterministic amplitude, ω_0 is a deterministic frequency, ϕ is a uniformly distributed random phase, and $w[n]$ is zero mean Gaussian noise with variance $\sigma_w^2 = 1$.

(a) Determine the corresponding autocorrelation sequence $r_x(k)$.

(b) Find the power spectral density $P_x(\omega)$, and draw a plot of $P_x(\omega)$.

(c) How can you estimate the frequency ω_0 from $P_x(\omega)$? What are two issues that complicate or limit this in practice?

(d) Find the autocorrelation matrix $\mathbf{R}_x$ of size 5×5.

(e) What are the eigenvalues of this matrix? What information do they give us on the signal?

(f) How can the frequency ω_0 be estimated from $\mathbf{R}_x$?

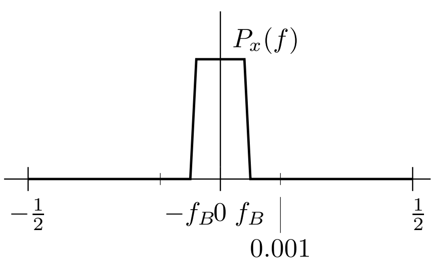
(g) In practice, we only have an estimate $\hat{\mathbf{R}}_x$. What is a suitable estimate $\hat{\mathbf{R}}_x$? Is the resulting frequency estimate consistent?

Question 2 (8 points)

Suppose we wish to determine the bandwidth of a slowly varying stochastic process $x(n)$. An example is when we wish to determine the Doppler bandwidth f_B, and the speed, of a mobile terminal in a wireless communication system. In this case, the power spectrum $P_x(f)$ is nonzero only for frequencies up to f_B, i.e. it satisfies

$$P_x(f) = 0, \quad f_B < |f| \leq 1/2\,.$$

We do not know f_B exactly, but we do know that $0 < f_B < 10^{-3}$.



We choose to compute an estimate $\hat{P}_x(f)$ of $P_x(f)$ from the data $x(n)$, plot $\hat{P}_x(f)$, and read the value of f_B from the plot.

(a) Suppose that $\hat{P}_x(f)$ is obtained by Bartlett's method with a window length of L. Suggest an appropriate value for L so that we can clearly read the value of f_B from our plot of $\hat{P}_x(f)$ (e.g., such that we reach a resolution of 10% of the maximum bandwidth). You may assume that the number of data samples is large, i.e., $N \gg L$.

(b) What is the variance of $\hat{P}_x(f)$ obtained using Bartlett's method?

(c) Suppose that we wish to achieve that the variance of $\hat{P}_x(f)$ is 10% of $P_x^2(f)$. How many samples of the signal are needed, and how is the spectrum to be estimated?

(d) If we wish to reduce the value of L, and the length of the FFTs used in Bartlett's method, we could downsample $x(n)$ before estimating the spectrum. The downsampled signal is $y(n) = x(nD)$, with D a positive integer.

The spectrum of the dowsampled signal $y(n) = x(nD)$ satisfies

$$P_y(f) = \frac{1}{D} P_x(f/D) \tag{1}$$

as long as aliasing does not occur.

How large can we choose the downsampling factor D without losing the possibility to estimate f_B from the spectrum of $y(n)$?

What is after downsampling a suitable choice for L to reach again a resolution of 10% of the maximum bandwidth?

Question 3 (12 points)

One problem in spectrum estimation concerns the issue whether or not a finite length sequence,

$$r_x(0), r_x(1), r_x(2), r_x(3), \ldots, r_x(p)$$

can be extended or extrapolated into a legitimate autocorrelation sequence in such a way that

$$P_x(e^{j\omega}) = \sum_{k=-\infty}^{\infty} r_x(k) e^{-jk\omega} \tag{2}$$

is a valid power spectrum. This is also known as the extendibility problem. To make it more concrete, is it possible to find values of $r_x(k)$ for $|k| > p$ such that $P_x(e^{j\omega})$ in (2) is a non-negative real function of ω?

(a) When is a $(p+1) \times (p+1)$ matrix $\mathbf{R}_x$ a valid autocorrelation matrix? How can you describe the extendibility problem in terms of conditions on $\mathbf{R}_x$?

(b) Alternatively, what is the condition on the p reflection coefficients, obtained from the Levinson recursion, which guarantees that $r_x(0), r_x(1), r_x(2), \ldots, r_x(p)$ are $p+1$ valid samples from an autocorrelation sequence?

(c) How can you extend the list of reflection coefficients in such a way that the related autocorrelation sequence is a legitimate autocorrelation sequence? Is the extension unique?

(d) Use the procedure developed in (b) and (c) to determine constraints on a and b that are required for the sequence

$$r_x(0) = 1, \quad r_x(1) = a, \quad r_x(2) = b$$

to be an extendible autcorrelation sequence.

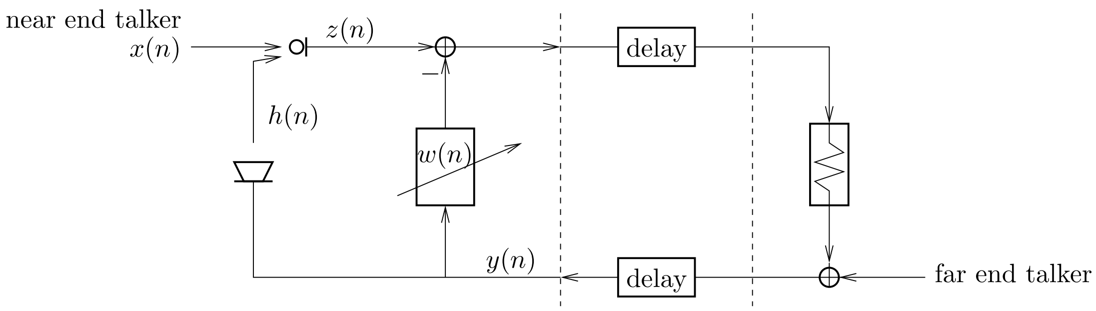
(e) Give a possible extension for $a = 0.3$ and $b = 0.9$, and explain how you obtain this extension.

Question 4 (10 points)

Consider the following echo cancellation problem for a hands free telephone system. The signal of interest that is picked up by the microphone is $x(n)$ (the near end signal) whereas the signal that is coming out of the speakers from the other end is $y(n)$ (the far end signal). Hence, the overall signal that is picked up by the microphone is given by

$$z(n) = x(n) + h(n) \star y(n) = x(n) + \sum_{l=0}^{L} h(l)y(n-l),$$

where $h(n)$ represents the propagation channel from the speaker to the microphone. Sending $z(n)$ to the other end is not a good idea since it will cause an echo. Hence, the problem is to find an estimate of $x(n)$ given only the signals $y(n)$ and $z(n)$, which are the only two signals known to the hands free telephone system, and which are assumed to be uncorrelated.



We will try to solve this echo cancellation problem by optimally estimating $z(n)$ from $y(n)$ using a linear filter $w(n)$.

(a) Derive the Wiener-Hopf equations for a linear filter $w(n)$ of order p such that if we apply this filter $w(n)$ to $y(n)$, we obtain the best possible estimate for $z(n)$ in the mean-square error sense.

Note that you have to give a derivation here and not simply copy equations from the book.

(b) Prove that when the filter order p is larger or equal to L, i.e., $p \geq L$, the optimal filter $w(n)$ is given by $h(n)$ padded with $p - L$ zeros at the end. In that case, also give an expression for the minimum mean-square error between $z(n)$ and its estimate.

Hint: Use the fact that $x(n)$ and $y(n)$ are uncorrelated.

(c) How can you use this optimal filter to estimate the near end signal $x(n)$? Can $x(n)$ be recovered exactly; or are there conditions under which this is possible?

(d) What is an LMS-type adaptive algorithm to estimate the filter coefficients $w(n)$?