Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Circuits and Systems Group

EE 4C03 STATISTICAL DIGITAL SIGNAL PROCESSING AND MODELING

10 November 2017, 13:30–16:30

Open book exam: copies of the book by Hayes and the course slides allowed. No other materials allowed.

This exam has four questions (40 points)

Question 1 (8 points)

Suppose we are given a linear shift-invariant system with system function

$$H(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{3}z^{-1}}$$

The system is excited by a zero mean noise process $x(n)$ with autocorrelation sequence

$$r_x(k) = \left(\frac{1}{2}\right)^{|k|}$$

(a) What is the power spectrum $P_x(z)$ of the input signal $x(n)$.

(b) What is the power spectrum $P_y(z)$ of the output signal $y(n)$.

(c) What is the autocorrelation sequence $r_y(k)$ of $y(n)$.

Hint: see Hayes, page 17 (table 2.4) for some useful z-transform pairs.

(d) Suppose we are given only $r_x(k)$ and $r_y(k)$, would it be possible to compute the corresponding filter $H(z)$ uniquely? (Explain.)

Solution

(a) Apply the z transform to $r_x(k)$:

$$P_x(z) = \frac{3/4}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{2}z)}\,.$$

(b)

$$P_y(z) = H(z)H^*(1/z^*)P_x(z) = \frac{3/4}{(1 - \frac{1}{3}z^{-1})(1 - \frac{1}{3}z)}\,.$$

(c) Use the z-transform pair

$$(\frac{1}{3})^{|k|} \quad \leftrightarrow \quad \frac{8/9}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{3}z)}.$$

This gives

$$r_y(k) = \frac{27}{32}(\frac{1}{3})^{|k|}$$

(d) From $P_y(z) = P_x(z)H(z)H^*(1/z^*)$ it is clear that if $P_x(z)$ and $P_y(z)$ are known, we can compute

$$H(z)H^*(1/z^*) = \frac{P_y(z)}{P_x(z)}.$$

Computing $H(z)$ from this expression is known as "spectral factorization" and is not unique. All poles and zeros occur in conjugate reciprocal pairs (if p is a pole, then $1/p*$ is also a pole, and likewise for zeros). We can assign all stable poles (within the unit circle) to $H(z)$ so that it is stable, but there is no such requirement for the zeros (unless we pose a minimum phase requirement). Moreover, we can always introduce a delay z^{-1} in $H(z)$ without modifying $H(z)H^*(1/z^*)$: delays are not identifiable.

Question 2 (10 points)

We want to model a signal $x(n)$ using an all-pole model of the form

$$H(z) = \frac{b(0)}{1 + a(N)z^{-N}}$$

where N is a given positive integer. We do this by minimizing the Prony error $\mathcal{E} = \sum_{n=0}^{\infty} |e(n)|^2$.

(a) In the Prony technique, how is the error $e(n)$ defined, for this particular case?

(b) Derive and solve the normal equations that result when minimizing the Prony error.

(c) Derive an expression for the minimum error.

(d) It is claimed that this model is suitable for modeling quasi-periodic signals (such as speech vowels). Explain why this is the case. Also explain the choice of N.

(e) If $|a(N)| \to 1$, then $H(z)$ has poles close to the unit circle. What type of signals does $H(z)$ model in this case (and why)?

Solution

(a) For an all-pole model, the error sequence is $e(n) = x(n) * a_p(n)$, where $a_p(n) \leftrightarrow A_p(z) = 1 + a(N)z^{-N}$. Thus

$$e(n) = x(n) + a(N)x(n-N)$$

(b) The normal equations are obtained by taking the derivative of the error function to the unknown coefficients of $a_p(n)$, and setting it to zero. Here:

$$\frac{d\mathcal{E}}{da(N)} = \sum_{n=0}^{\infty} 2e(n)x(n-N) = 0.$$

Substitute $e(n) = x(n) + a(N)x(n-N)$ to obtain

$$a(N)\sum_{n=0}^{\infty}[x(n-N)]^2 = -\sum_{n=0}^{\infty} x(n)x(n-N)$$

Define $r(k) = \sum_{n=0}^{\infty} x(n)x(n-k)$. Then the normal equations is the single equation

$$a(N)r(0) = -r(N)\,.$$

Solving for $a(N)$ gives

$$a(N) = -\frac{r(N)}{r(0)}$$

(c) The error function is

$$\begin{aligned}\mathcal{E} &= \textstyle\sum_{n=0}^{\infty} |e(n)|^2 = \sum_{n=0}^{\infty}[x(n) + a(N)x(n-N)][x(n) + a(N)x(n-N)] \\ &= r(0)(1 + [a(N)]^2) + 2a(N)r(N)\end{aligned}$$

(where we assumed real signals and coefficients). Insert the solution $a(N)$ to obtain

$$\{\mathcal{E}\}_{\min} = r(0)\left(1 + \frac{r^2(N)}{r^2(0)}\right) - 2\frac{r^2(N)}{r(0)} = r(0) - \frac{r^2(N)}{r(0)}$$

Alternative derivation: use [Hayes, Table 4.2]

$$\{\mathcal{E}\}_{\min} = r_x(0) + \sum_{k=1}^{N} a_p(k)r_x(k) = r(0) - a(N)r(N) = r(0) - \frac{r^2(N)}{r(0)}$$

(d) Suppose we excite this filter with white noise $e(n)$. The filter is an AR process, the generated signal is $x(n) = -a(N)x(n-N) + e(n)$. Thus, $x(n)$ is correlated to $x(n-N)$. If $|a(N)| < 1$, the correlation is not 100%, thus the signal is not perfectly periodic (there is an "innovation" by $e(n)$).

N is the period of the quasi-periodic signal. This corresponds to the peak that should be visible in the correlation sequence $r_x(k)$.

(e) The generated sequence is now perfectly periodic. Note that the filter is marginally stable and to have a bounded output the random input sequence should be finite length (e.g. N random samples and then zeros).

Another way to interprete this is in the frequency domain. The power spectrum generated by the filter is $P_x(z) = H(z)H^*(1/z^*)$. Note that $H(z)$ has N poles at $z = (a(N))^(1/N)e^{j(\pi/N+2\pi k/N}$, $k = 0, \ldots, N-1$.

For $a(N)$ close to the unit circle, the poles of $H(z)$ are close to the unit circle. The spectrum of $x(n)$ then has strong spikes, resulting in a sum of N uncorrelated harmonic processes (cosines with random initial phases) with frequencies that are multiples of $1/N$. Again, N is the period.

Question 3 (11 points)

We have measured a data sequence $x(n)$, $n = 0, \cdots, N-1$. The model for $x(n)$ is a single sinusoid in noise,

$$x(n) = s(n) + w(n)\,, \qquad s(n) = A\sin(\omega_0 n + \phi)$$

where A is an amplitude, ω_0 is a carrier frequency, and ϕ is a random phase. Both A and ω_0 are unknown deterministic constants, and ϕ is modeled as uniformly distributed over the interval $[-\pi, \pi]$. The measurements are contaminated by independent zero mean additive white Gaussian noise $w(n)$ with known variance σ_w^2 per sample.

We wish to detect the presence of $s(n)$ using Bartlett's method to estimate the power spectrum of $x(n)$. The signal is detected if we see a "significant" spike in the spectrum estimate. To make this explicit, we define that the spike should be larger than 2 times the standard deviation of the spectrum estimate in case the sinusoidal signal was not present.

(a) For this model, is $x(n)$ wide sense stationary? If so, determine an expression for its power spectrum $P_x(e^{j\omega})$ in terms of the given parameters.

(b) Give a (detailed) expression for the expected value of Bartlett's power spectrum estimate, $E\{\hat{P}_B(e^{j\omega})\}$.

Illustrate this with a graph that shows both $P_x(e^{j\omega})$ and $E\{\hat{P}_B(e^{j\omega})\}$. Is Bartlett's power spectrum estimate an unbiased estimate?

(c) What is the variance of the spectrum estimate in case the sinusoidal signal is not present?

(d) Suppose $N = 1000$. We can choose the number of segments K and the length of each segment L in several ways. Discuss possible options and their consequences on resolution and sensitivity of the spectrum estimate. (Sensitivity relates to the weakest possible sinusoid that can be detected.)

(e) We would like to optimize the sensitivity of the method, i.e., enable detection of the weakest possible sinusoid. Determine L and K.

(f) What is the corresponding resolution of the spectrum estimate?

Solution

(a) It is WSS. The autocorrelation sequence is (Hayes p.96)

$$r_x(k) = \frac{1}{2}A^2\cos(k\omega_0) + \sigma_w^2\delta(k)$$

and the corresponding power spectrum is

$$P_x(e^{j\omega}) = \frac{1}{2}\pi A^2[u_0(\omega-\omega_0) + u_0(\omega+\omega_0)] + \sigma_w^2$$

where $u_0(\omega)$ is a unit-power delta-spike in ω-domain (Hayes p.13).

(b) See Hayes p.413 and also compare to p.399:

$$E\{\hat{P}_B(e^{j\omega})\} = \frac{1}{2\pi} P_x(e^{j\omega}) * W_B(e^{j\omega})$$

where the expression for $P_x(e^{j\omega})$ is given above, and $W_B()$ is the Fourier transform of a Bartlett window $w_B(k)$ that extends from $-L$ to L, so that (Hayes p.409)

$$W_B(e^{j\omega}) = \frac{1}{L}\left[\frac{\sin(L\omega/2)}{\sin(\omega/2)}\right]^2$$

Note that W_B has unit energy, i.e. $\int W_B(e^{j\omega})d\omega = 2\pi$; its peak value is L and the width of the peak is $\Delta\omega = 2\pi/L$. Inserting in the preceding expression, we obtain

$$E\{\hat{P}_B(e^{j\omega})\} = \frac{1}{4}A^2[W_B(e^{j(\omega-\omega_0)}) + W_B(e^{j(\omega+\omega_0)})] + \sigma_w^2$$

The plot resembles Hayes fig. 8.5 but the main peak has height $\frac{1}{4}A^2 \cdot L + \sigma_w^2$. (Note error in Hayes' figure where the height is labeled wrong.)

This is not an unbiased estimate. However, it is asymptotically (large L) unbiased.

(c) Table 8.4:

$$\text{Var}\{\hat{P}_B(e^{j\omega})\} \approx \frac{1}{K} P_x^2(e^{j\omega}) = \frac{1}{K}\sigma_w^4$$

(d) We could choose K small and L large, e.g. $K = 10$, $L = 100$. Alternatively, we could choose K large and L small, e.g., $K = 100$, $L = 10$.

In the first case, we obtain a spectrum estimate with $L = 100$ bins, resolution is $0.98 \cdot 2\pi/100$, peak of the sinusoid is expected at $\frac{1}{4}A^2 \cdot 100$. The variance of the estimate is $\frac{1}{10}\sigma_w^4$. The peak can be detected if it is above twice the standard deviation, i.e. if

$$\frac{1}{4}A^2 \cdot 100 > \frac{2}{\sqrt{10}}\sigma_w^2 \qquad \Rightarrow \qquad A^2 > \frac{8}{100\sqrt{10}}\sigma_w^2$$

In the second case, we obtain resolution $0.98 \cdot 2\pi/10$ (much poorer), peak of sinusoid at $\frac{1}{4}A^2 \cdot 10$, variance $\frac{1}{100}\sigma_w^4$, detection if

$$\frac{1}{4}A^2 \cdot 10 > \frac{2}{\sqrt{100}}\sigma_w^2 \qquad \Rightarrow \qquad A^2 > \frac{8}{10\sqrt{100}}\sigma_w^2$$

Perhaps surprisingly, this is less sensitive (for the same σ_w, a higher amplitude A is required for detection).

In conclusion, both resolution and sensitivity benefit from a large L and small K. (Note that we had a sinusoidal signal with an infinitely narrow passband. If the signal is modulated and has a bandwidth B, the result is valid only as long as $\Delta\omega > B$, i.e., the source is not resolved.)

(e) We best choose $K = 1$ and $L = 1000$.

(f) $\Delta\omega = 0.98 \cdot 2\pi/L = 0.98 \cdot 2\pi/1000$.

Question 4 (11 points)

We would like to estimate a process $d(n)$ from noisy observations,

$$x(n) = d(n) + v(n)$$

where $v(n)$ is white noise with variance $\sigma_v^2 = 1$, and $d(n)$ is a wide-sense stationary process with the first four values of the autocorrelation sequence given by

$$\mathbf{r}_d = [1.5,\ 0,\ 1.0,\ 0]^T$$

Assume that $d(n)$ and $v(n)$ are uncorrelated. Our goal is to design an FIR filter to reduce the noise in $d(n)$. Hardware constraints, however, limit the filter to only three nonzero coefficients in $W(z)$.

(a) What would be an example signal $d(n)$ that satisfies this correlation model?

(b) Derive the Wiener-Hopf equations and their solution to determine the optimal three-multiplier causal filter

$$W(z) = w(0) + w(1)z^{-1} + w(2)z^{-2}$$

for estimating $d(n)$ from $x(n)$. Also evaluate the mean-square error $E(|d(n) - \hat{d}(n)|^2)$.

Hint: you may want to use the following matrix identity:

$$\begin{bmatrix} p & q \\ q & p \end{bmatrix}^{-1} = \frac{1}{p^2 - q^2} \begin{bmatrix} p & -q \\ -q & p \end{bmatrix}$$

(c) Repeat (b) for the noncausal FIR filter

$$W(z) = w(-1)z + w(0) + w(1)z^{-1}$$

(d) Can you suggest a way to reduce the mean-square error below that obtained for the filters designed in parts (b) and (c), but without using more than three filter coefficients?

(e) Suppose that we have measured $x(n)$ and know the noise power σ_v^2 but do not know the correlation sequence $r_d(k)$. How can we implement the optimal filter of part (b) in practice?

Solution

(a) The signal can be of the form

$$d(n) = a\,s(n) + b\,s(n-2)$$

where $s(n)$ is zero mean unit variance white noise. The coefficients a and b are determined from $r_d(0) = a^2 + b^2 = 1.5$ and $r_d(2) = a\,b = 1.0$.

(b) The Wiener-Hopf equations for the optimal three-multiplier filter

$$W(z) = w(0) + w(1)z^{-1} + w(2)z^{-2}$$

for estimating $d(n)$ are

$$\begin{bmatrix} r_x(0) & r_x(1) & r_x(2) \\ r_x(1) & r_x(0) & r_x(1) \\ r_x(2) & r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \\ w(2) \end{bmatrix} \begin{bmatrix} r_{dx}(0) \\ r_{dx}(1) \\ r_{dx}(2) \end{bmatrix}$$

With $r_x(k) = r_d(k) + r_v(k)$ and $r_{dx}(k) = r_x(k)$, the equations become

$$\begin{bmatrix} 2.5 & 0 & 1.0 \\ 0 & 2.5 & 0 \\ 1.0 & 0 & 2.5 \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \\ w(2) \end{bmatrix} \begin{bmatrix} 1.5 \\ 0 \\ 1.0 \end{bmatrix}$$

The solution is

$$\begin{bmatrix} w(0) \\ w(1) \\ w(2) \end{bmatrix} = \begin{bmatrix} 0.5238 \\ 0 \\ 0.1905 \end{bmatrix}$$

The mean square error is

$$E(|d(n) - \hat{d}(n)|^2 = r_d(0) - \sum_{k=0}^{2} w(k) r_{dx}(k) = 1.5 - 0.5238 \cdot 1.5 - 0.1905 = 0.5238$$

(c) For the noncausal FIR filter

$$W(z) = w(-1)z + w(0) + w(1)z^{-1}$$

the Wiener-Hopf equations are

$$\begin{bmatrix} r_x(0) & r_x(1) & r_x(2) \\ r_x(1) & r_x(0) & r_x(1) \\ r_x(2) & r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} w(-1) \\ w(0) \\ w(1) \end{bmatrix} \begin{bmatrix} r_{dx}(-1) \\ r_{dx}(0) \\ r_{dx}(1) \end{bmatrix}$$

Substituting, we find

$$\begin{bmatrix} 2.5 & 0 & 1.0 \\ 0 & 2.5 & 0 \\ 1.0 & 0 & 2.5 \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \\ w(2) \end{bmatrix} \begin{bmatrix} 0 \\ 1.5 \\ 0 \end{bmatrix}$$

The solution is

$$\begin{bmatrix} w(-1) \\ w(0) \\ w(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0.6 \\ 0 \end{bmatrix}$$

The mean square error is

$$E(|d(n) - \hat{d}(n)|^2 = r_d(0) - \sum_{k=-1}^{1} w(k) r_{dx}(k) = 0.6$$

(d) In both previous cases, we obtain some zero filter coefficients, because $d(n \pm 1)$ is uncorrelated to $d(n)$. Therefore, there is no reason to use the corresponding samples $x(n \pm 1)$. A better estimator uses either

$$W(z) = w(0) + z^{-2}w(2) + z^{-4}w(4)$$

or

$$W(z) = z^{2}w(-2) + w(0) + z^{-2}w(2)$$

(Both cases would have to be tested to see which one gives a smaller error.)

(e) From $x(n)$ we can estimate $r_x(k)$. The problem is that we don't know $r_d(k)$. However, we know $r_x(k) = r_d(k) + \sigma_v^2\delta(k)$; thus we can derive $r_d(k)$ from this.