Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Circuits and Systems Group

EE4C03 STATISTICAL DIGITAL SIGNAL PROCESSING AND MODELING

10 November 2017, 13:30–16:30

Open book exam: copies of the book by Hayes and the course slides allowed. No other materials allowed.

This exam has four questions (40 points)

Question 1 (8 points)

Suppose we are given a linear shift-invariant system with system function

$$H(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{3}z^{-1}}$$

The system is excited by a zero mean noise process $x(n)$ with autocorrelation sequence

$$r_x(k) = \left(\frac{1}{2}\right)^{|k|}$$

(a) What is the power spectrum $P_x(z)$ of the input signal $x(n)$.

(b) What is the power spectrum $P_y(z)$ of the output signal $y(n)$.

(c) What is the autocorrelation sequence $r_y(k)$ of $y(n)$.

Hint: see Hayes, page 17 (table 2.4) for some useful z-transform pairs.

(d) Suppose we are given only $r_x(k)$ and $r_y(k)$, would it be possible to compute the corresponding filter $H(z)$ uniquely? (Explain.)

Question 2 (10 points)

We want to model a signal $x(n)$ using an all-pole model of the form

$$H(z) = \frac{b(0)}{1 + a(N)z^{-N}}$$

where N is a given positive integer. We do this by minimizing the Prony error $\mathcal{E} = \sum_{n=0}^{\infty} |e(n)|^2$.

(a) In the Prony technique, how is the error $e(n)$ defined, for this particular case?

(b) Derive and solve the normal equations that result when minimizing the Prony error.

(c) Derive an expression for the minimum error.

(d) It is claimed that this model is suitable for modeling quasi-periodic signals (such as speech vowels). Explain why this is the case. Also explain the choice of N.

(e) If $|a(N)| \to 1$, then $H(z)$ has poles close to the unit circle. What type of signals does $H(z)$ model in this case (and why)?

Question 3 (11 points)

We have measured a data sequence $x(n)$, $n = 0, \cdots, N-1$. The model for $x(n)$ is a single sinusoid in noise,

$$x(n) = s(n) + w(n)\,, \qquad s(n) = A\sin(\omega_0 n + \phi)$$

where A is an amplitude, ω_0 is a carrier frequency, and ϕ is a random phase. Both A and ω_0 are unknown deterministic constants, and ϕ is modeled as uniformly distributed over the interval $[-\pi, \pi]$. The measurements are contaminated by independent zero mean additive white Gaussian noise $w(n)$ with known variance σ_w^2 per sample.

We wish to detect the presence of $s(n)$ using Bartlett's method to estimate the power spectrum of $x(n)$. The signal is detected if we see a "significant" spike in the spectrum estimate. To make this explicit, we define that the spike should be larger than 2 times the standard deviation of the spectrum estimate in case the sinusoidal signal was not present.

(a) For this model, is $x(n)$ wide sense stationary? If so, determine an expression for its power spectrum $P_x(e^{j\omega})$ in terms of the given parameters.

(b) Give a (detailed) expression for the expected value of Bartlett's power spectrum estimate, $E\{\hat{P}_B(e^{j\omega})\}$.

Illustrate this with a graph that shows both $P_x(e^{j\omega})$ and $E\{\hat{P}_B(e^{j\omega})\}$. Is Bartlett's power spectrum estimate an unbiased estimate?

(c) What is the variance of the spectrum estimate in case the sinusoidal signal is not present?

(d) Suppose $N = 1000$. We can choose the number of segments K and the length of each segment L in several ways. Discuss possible options and their consequences on resolution and sensitivity of the spectrum estimate. (Sensitivity relates to the weakest possible sinusoid that can be detected.)

(e) We would like to optimize the sensitivity of the method, i.e., enable detection of the weakest possible sinusoid. Determine L and K.

(f) What is the corresponding resolution of the spectrum estimate?

Question 4 (11 points)

We would like to estimate a process $d(n)$ from noisy observations,

$$x(n) = d(n) + v(n)$$

where $v(n)$ is white noise with variance $\sigma_v^2 = 1$, and $d(n)$ is a wide-sense stationary process with the first four values of the autocorrelation sequence given by

$$\mathbf{r}_d = [1.5,\ 0,\ 1.0,\ 0]^T$$

Assume that $d(n)$ and $v(n)$ are uncorrelated. Our goal is to design an FIR filter to reduce the noise in $d(n)$. Hardware constraints, however, limit the filter to only three nonzero coefficients in $W(z)$.

(a) What would be an example signal $d(n)$ that satisfies this correlation model?

(b) Derive the Wiener-Hopf equations and their solution to determine the optimal three-multiplier causal filter

$$W(z) = w(0) + w(1)z^{-1} + w(2)z^{-2}$$

for estimating $d(n)$ from $x(n)$. Also evaluate the mean-square error $E(|d(n) - \hat{d}(n)|^2)$.

Hint: you may want to use the following matrix identity:

$$\begin{bmatrix} p & q \\ q & p \end{bmatrix}^{-1} = \frac{1}{p^2 - q^2} \begin{bmatrix} p & -q \\ -q & p \end{bmatrix}$$

(c) Repeat (b) for the noncausal FIR filter

$$W(z) = w(-1)z + w(0) + w(1)z^{-1}$$

(d) Can you suggest a way to reduce the mean-square error below that obtained for the filters designed in parts (b) and (c), but without using more than three filter coefficients?

(e) Suppose that we have measured $x(n)$ and know the noise power σ_v^2 but do not know the correlation sequence $r_d(k)$. How can we implement the optimal filter of part (b) in practice?