Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Circuits and Systems Group

EE 4C03 STATISTICAL DIGITAL SIGNAL PROCESSING AND MODELING

10 November 2016, 13:30–16:30

Open book exam: copies of the book by Hayes and the course slides allowed. No other materials allowed.

This exam has four questions (40 points).

Question 1 (9 points)

Consider the complex random process

$$x(n) = Ae^{j(\omega_0 n + \phi)} + w(n)$$

where $w(n)$ is a zero mean white Gaussian noise random process with variance σ_w^2. For each of the following cases,

– find the mean and the autocorrelation sequence of $x(n)$;

– if the process is wide sense stationary (WSS), find the power spectrum.

(a) A is a Gaussian random variable with zero mean and variance σ_A^2, and both ω_0 and ϕ are constants.

(b) ϕ is uniformly distributed over the interval $[-\pi, \pi]$ and both A and ω_0 are constants.

(c) ω_0 is a random variable that is uniformly distributed over some interval $[\Omega_0 - \Delta, \Omega_0 + \Delta]$, and both A and ϕ are constants.

(d) ω_0 is a random variable that is uniformly distributed over some interval $[\Omega_0 - \Delta, \Omega_0 + \Delta]$, ϕ is uniformly distributed over the interval $[-\pi, \pi]$, and A is a constant.

Hint: you may need this DTFT pair:

$$x(n) = \frac{\sin(\Delta n)}{\pi n} \quad \leftrightarrow \quad X(\omega) = \begin{cases} 1, & |\omega| < \Delta \\ 0, & \text{elsewhere} \end{cases}$$

Question 2 (10 points)

Suppose that $x(n)$ is a wide-sense stationary process with autocorrelation sequence $r_x(n)$ and our goal is to model $x(n)$ using an optimal AR (all-pole) model with a system transfer function of the form

$$H(z) = \frac{b(0)}{1 + \sum_{k=1}^{p} a_p(k) z^{-k}} ,$$

where p is the model order. We consider the Levinson-Durbin recursion and the corresponding FIR lattice filter.

(a) Write down the Yule-Walker equations for this scenario.

(b) What is the computational complexity (order of magnitude) of the Levinson method as a function of the filter order p? And what is the complexity of solving the Yule-Walker equations directly?

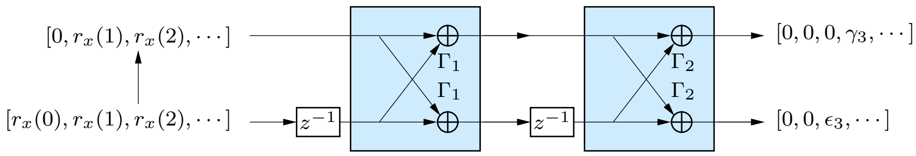
(c) Assume that the autocorrelation function of $x(n)$ is given by

$$r_x(-3) = 0,\ r_x(-2) = 1,\ r_x(-1) = 2,\ r_x(0) = 3,\ r_x(1) = 2,\ r_x(2) = 1,\ r_x(3) = 0.$$

Determine the reflection coefficients Γ_j, the model parameters $a_j(k)$ and the modeling errors ϵ_j for $j = 1, 2$.

(d) Conversely, given the reflection coefficients, it should be possible to recover the autocorrelation sequence.

Recall that the Schur recursion resulted in a FIR lattice filter that showed how the reflection coefficients are obtained from the autocorrelation sequence:



Starting from this, derive a lattice filter implementation that, given the reflection coefficients and $r_x(0)$, generates the rest of autocorrelation sequence at one of its outputs. (Specify also the input of the filter.)

Now suppose that $x(n)$ is a wide-sense stationary process and our goal is to predict $x(n)$ one step ahead. In other words, we want to estimate $x(n+1)$ from $x(n)$, $x(n-1)$, …, $x(n-p)$ using a linear filter $w(n)$ of order p, i.e., $\hat{x}(n+1) = w(0)x(n) + w(1)x(n-1) + \cdots + w(p)x(n-p)$.

(e) Write down the Wiener-Hopf equations for this scenario (without solving them).

(f) How is the solution for the filter coefficients $w(n)$ related to the optimal all-pole model for the random process $x(n)$? Write down the system transfer function for this all-pole model using the filter coefficients $w(n)$.

Question 3 (12 points)

Consider a signal $s(n)$ that is obtained by filtering zero-mean white noise with variance 1. More specifically, assume that

$$s(n) = v(n) + v(n-1),$$

with $v(n)$ zero-mean white noise with variance 1. Also assume that this filtering process is corrupted by noise. In other words, we can only measure the signal

$$x(n) = s(n) + w(n),$$

with $w(n)$ also zero-mean white noise with variance 1.

(a) Determine the unit sample response of the filter with input $v(n)$ and output $s(n)$. Also derive the frequency response of that filter and make a plot of the magnitude of the frequency response. What type of filter is this, low-pass, band-pass, or high-pass?

(b) Derive the autocorrelation function $r_s(k)$ of $s(n)$, the autocorrelation function $r_x(k)$ of $x(n)$, and the cross-correlation function $r_{sx}(k)$ between $s(n)$ and $x(n)$.

(c) Determine the unit sample response of a FIR filter of length 2 which optimally estimates the signal $s(n+1)$ from $x(n)$. Also compute the related minimal estimation error.

Hint: The inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is given by* $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

(d) Repeat (c) for a FIR filter of length 3. So again determine the unit sample response and compute the estimation error. Is the filter of length 3 better than the filter of length 2?

Hint: In a first step, try to use the last Wiener-Hopf equation to transform the system of three equations into a system of two equations. Then solve this as before.

(e) Write down the update equations of the normalized LMS (NLMS) algorithm to solve this estimation problem adaptively (you can either take the filter of length 2 or 3 to explain).

(f) Is there any difference between estimating $s(n+1)$ from $x(n)$ or estimating $x(n+1)$ from $x(n)$? Explain why there is a difference or why not.

Question 4 (9 points)

A continuous-time signal $x_a(t)$ is bandlimited to 5 kHz, i.e., $x_a(t)$ has a spectrum $X_a(f)$ that is zero for $|f| > 5$ kHz. Only 10 seconds of the signal has been recorded and is available for processing. We would like to estimate the power spectrum of $x_a(t)$ using the available data with a DFT algorithm, and it is required that the estimate has a resolution of at least 10 Hz. Suppose that we use Bartlett's method of periodogram averaging.

(a) What is the Nyquist rate for this signal? If the data is sampled at the Nyquist rate, what is the mimimum section length L that you may use to get the desired resolution?

(b) Using this minimum section length, with 10 seconds of data, how many sections are available for averaging?

(c) What is the variance of the resulting Bartlett's spectrum estimate?

(d) What happens to the variance if we sample at twice the Nyquist rate (still collecting 10 seconds of data and aiming for a resolution of 10 Hz)?

Is there a benefit in sampling faster than Nyquist?

(e) Now we sample at Nyquist rate, but consider a Welch power spectrum estimate with a Bartlett window and 50% overlap. If the estimate has a resolution of 10 Hz, then what is the variance of the Welch estimate?

Is there a benefit in using the Welch estimate compared to the Bartlett estimate? Does the choice of the window play any role?