Applied Convex Optimization

Circuit and Systems Group

Delft University of Technology

Tuesday 11th December, 2018

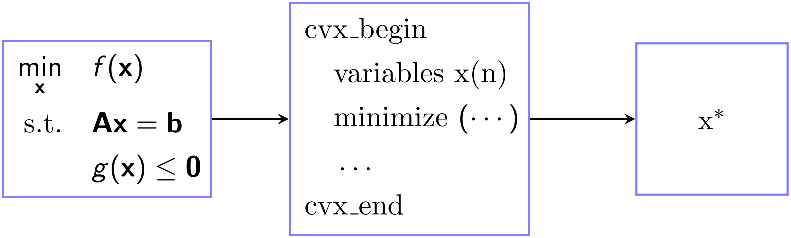
Outline

Next Subsection

1. First Half
   - CVX: A Convex Optimisation Toolbox
   - Convex Sets
   - Convex Functions
2. Second Half
   - Converting Convex Problems
   - Lagrange Duality

What is CVX?

- CVX is a modeling system for convex optimisation problems
- Website: http://cvxr.com/cvx



Structure of Convex Problems

Mathematically[1]

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & f_0(\mathbf{x}) \\
\text{s.t.} \quad & f_i(\mathbf{x}) \leq \mathbf{0}, \quad i = 1, \cdots, m \\
& h_j(\mathbf{x}) = 0, \quad j = 1, \cdots, p
\end{aligned}
$$

In CVX

```
cvx_begin
    variables x(n)
    minimize (f0(x))
    subject to
        f(x) <= 0
        A * x - b == 0
cvx_end
```

[1] f_0 and f_i must be convex and h_i must be affine.

Return Values

Upon exit, CVX sets the variables

- x - solution variables(s) x^*
- $\mathrm{cvx\_optval}$ - the optimal value p^*
- $\mathrm{cvx\_status}$ - solver status (Solved, Unbounded, Infeasible,$\cdots$)
- $\cdots$

Basic Example - LP

Optimization Problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

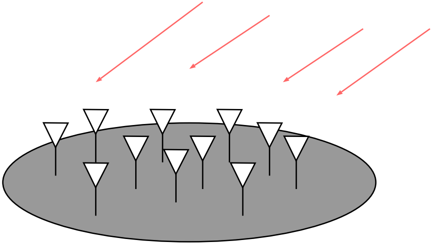
In CVX

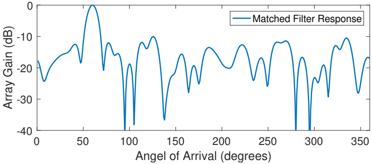
```
cvx_begin
    variables x(n)
    minimize (c' * x)
    subject to
        A * x - b == 0
        x >= 0
cvx_end
```

Basic Example - LP

Demo in Matlab

Practical Example





Beam Pattern Optimization

Given an arbitrary N element antenna array, design a configuration for the antennas such that

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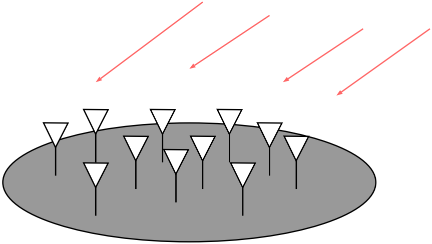
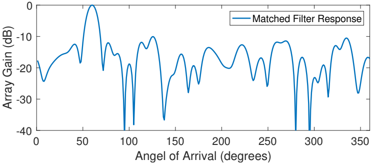


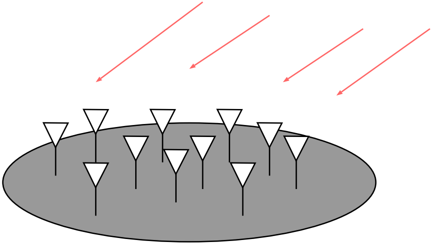


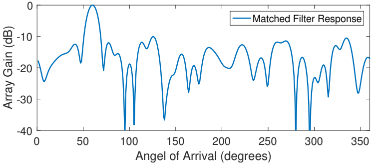
Beam Pattern Optimization

Given an arbitrary N element antenna array, design a configuration for the antennas such that

- The gain in a target direction is unity (target signal is preserved)

Practical Example





Beam Pattern Optimization

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- The worst case side lobe gain of the setup is minimized

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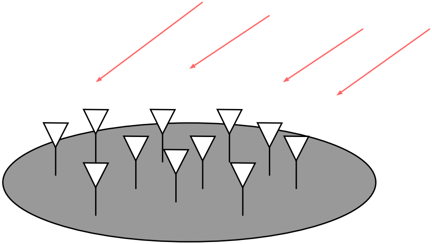

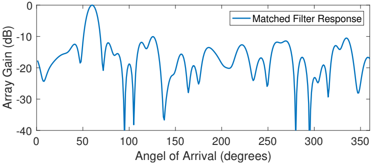

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Given an arbitrary N element antenna array, design a configuration for the antennas such that

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The task is to design a set of weights $\mathbf{w}$ to meet these performance requirements.

Beam Pattern Optimization

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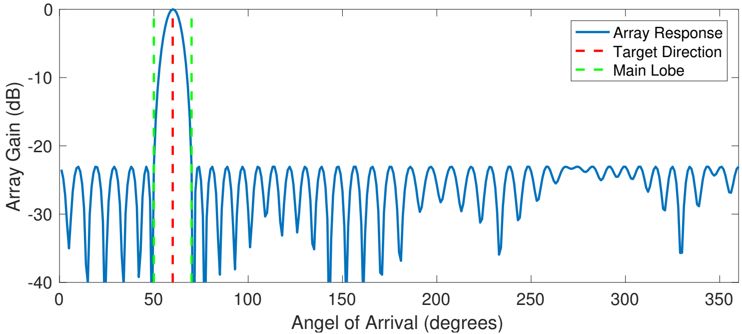
```
cvx_begin
  variable w(n) complex
  minimize (norm(A_sl' * w,Inf))
  subject to
    a_tar' * w == 1
cvx_end
```

Optimized Beam Responses



Beam Pattern Optimization

Demo in Matlab

Next Subsection

Proving Set Convexity

Methods

Proving Set Convexity

Methods

- Definition - C is convex if and only if $\forall\ x_1, x_2 \in C,\ \theta \in \{0, 1\},\ \theta x_1 + (1 - \theta) x_2 \in C$.

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Example Problem: Quadratic Constraint Sets

Example 1: Quadratic Constraint Set

Show that the quadratic constraint set

$$C = \{\mathbf{x} \mid \mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{q}^T\mathbf{x} + c \leq 0\}$$

is convex if $\mathbf{Q} \succeq \mathbf{0}$.

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Three Alternatives

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- Relationship with known convex sets

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Three Alternatives

- Properties of convex sets
- Relationship with known convex sets
- Using properties of convex functions

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Intersection With Arbitrary Line

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Intersection With Arbitrary Line

Recall that a set is convex if and only if its intersection with an arbitrary line is convex.

Define the arbitrary line $\mathbf{b} + t\mathbf{v}$ where $t \in \mathbb{R}$. By substitution

$$\begin{aligned}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{q}^T\mathbf{x} + c &= (\mathbf{b} + t\mathbf{v})^T\,\mathbf{Q}\,(\mathbf{b} + t\mathbf{v}) + \mathbf{q}^T\,(\mathbf{b} + t\mathbf{v}) + c \\ &= \alpha t^2 + \beta t + \gamma\end{aligned}$$

where $\alpha = \mathbf{v}^T\mathbf{Q}\mathbf{v}$, $\beta = \mathbf{b}^T\mathbf{Q}\mathbf{v} + \mathbf{q}^T\mathbf{v}$ and $\gamma = \mathbf{b}^T\mathbf{Q}\mathbf{b} + \mathbf{q}^T\mathbf{b} + c$.

If $\alpha \geq 0$, C is a simple ellipsoid and is convex. For $\alpha \geq 0 \forall \mathbf{v}$, $\mathbf{Q} \succeq 0$.

Example Problem: Quadratic Constraint Sets

Relationship With Euclidean Ball

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Relationship With Euclidean Ball

Recall that the Euclidean ball given by

$$\mathcal{E} = \{\mathbf{x} \mid (\mathbf{x} - \mathbf{x}_c)^T \mathbf{P}^{-1} (\mathbf{x} - \mathbf{x}_c) \leq 1\}$$

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Expanding the quadratic form

$$\mathcal{E} = \{\mathbf{x} \mid \mathbf{x}^T \mathbf{P}^{-1} \mathbf{x} - 2\mathbf{x}_c^T \mathbf{P}^{-1} \mathbf{x} + \mathbf{x}_c^T \mathbf{P}^{-1} \mathbf{x}_c \leq 1\}$$

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which can be rewritten in the form

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which can be rewritten in the form

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Thus convexity is proven by association with a known convex set.

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Convexity of Quadratic Functions

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Therefore, $\forall\ \mathbf{x}_1, \mathbf{x}_2 \in C,\ \theta \in \{0, 1\}$ it follows that

$$(\theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2)^T\,\mathbf{Q}\,(\theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2) \leq \theta\mathbf{x}_1^T\mathbf{Q}\mathbf{x}_1 + (1-\theta)\mathbf{x}_2^T\mathbf{Q}\mathbf{x}_2$$

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Therefore, we can show that

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$$\leq \theta\left(\mathbf{x}_1^T\mathbf{Q}\mathbf{x}_1 + \mathbf{q}^T\mathbf{x}_1 + c\right) + (1-\theta)\left(\mathbf{x}_2^T\mathbf{Q}\mathbf{x}_2 + \mathbf{q}^T\mathbf{x}_2 + c\right) \leq 0$$

such that C is a convex set.

Example Problem: Hyperbolic Constraint Sets

Example 2: Hyperbolic Sets

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Show that the hyperbolic constraint set

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Hint: If $a, b \geq 0$ and $\theta \in [0, 1]$, then $a^\theta b^{(1-\theta)} \leq \theta a + (1-\theta)b$.

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From the hint we know that

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Rearranging and applying the definition of C we get

$$\begin{aligned}(\theta x_{a,1} + (1-\theta)x_{b,1})(\theta x_{a,2} + (1-\theta)x_{b,2}) &\geq (x_{a,1}x_{a,2})^{\theta}(x_{b,1}x_{b,2})^{(1-\theta)} \\ &\geq 1^{\theta}1^{(1-\theta)} \\ &= 1\end{aligned}$$

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for all $\mathbf{x}_a,\ \mathbf{x}_b \in C$ and a $\theta \in [0,1]$.

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for all $\mathbf{x}_a$, $\mathbf{x}_b \in C$ and a $\theta \in [0,1]$. In this way, C is a convex set.

Next Subsection

Showing That a Function is Convex

Methods

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- Definition - f is convex if and only if
  $\forall\ x_1, x_2 \in \mathrm{dom}(f),\ \theta \in \{0, 1\},\ f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta)f(x_2)$.

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- 1st order condition - $\forall\, x_1, x_2 \in \mathrm{dom}(f), f(x_1) \geq f(x_2) + \nabla f(x_2)^T (x_1 - x_2)$

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- 1st order condition - $\forall\, x_1, x_2 \in \mathrm{dom}(f), f(x_1) \geq f(x_2) + \nabla f(x_2)^T(x_1 - x_2)$
- 2nd order condition - $\forall\, x_1 \in \mathrm{dom}(f),\ \nabla^2 f(x_1) \succ 0$

Showing That a Function is Convex

Methods

- Definition - f is convex if and only if $\forall\, x_1, x_2 \in \mathrm{dom}(f),\ \theta \in \{0,1\},\ f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta) f(x_2)$.
- 1st order condition - $\forall\, x_1, x_2 \in \mathrm{dom}(f), f(x_1) \geq f(x_2) + \nabla f(x_2)^T (x_1 - x_2)$
- 2nd order condition - $\forall\, x_1 \in \mathrm{dom}(f),\ \nabla^2 f(x_1) \succ 0$
- Composition rules - $f(x) = h \circ g(x)$ is convex if either h is convex, h' is non-decreasing and g is convex or h is convex, h' is non-increasing and g is concave.

Basic Example

Example Problem 1

Show that the function

$$f(x,t) = -\log(t^p - \|x\|_p^p)$$

is convex if where $p \geq 2$ and $\mathrm{dom}(f) = \{(x,t) | t > \|x\|_p\}$.

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Alternatively we can use convexity preserving composition rules.

Basic Example

Example Problem 1

Begin by noting that

$$\begin{aligned} f(x,t) &= -\log(t^p - \|x\|_p^p) \\ &= -\log(t^{p-1}) - \log\left(t - \frac{\|x\|_p^p}{t^{p-1}}\right) \\ &= -(p-1)\log(t) - \log\left(t - \frac{\|x\|_p^p}{t^{p-1}}\right). \end{aligned}$$

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The left hand term is a convex function.

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The left hand term is a convex function. The right hand term is the composition of a convex, non-increasing function and

$$t - \frac{\|x\|_p^p}{t^{p-1}} \geq 0.$$

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The left hand term is a convex function. The right hand term is the composition of a convex, non-increasing function and

$$t - \frac{\|x\|_p^p}{t^{p-1}} \geq 0.$$

To show convexity this term must be concave, i.e. we want to show that $\frac{\|x\|_p^p}{t^{p-1}}$ is convex.

Basic Example

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We can show convexity by noting a relationship with the known convex function $\frac{z}{s}$ i.e the perspective function.

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The perspective function is linear in z such that

$$\frac{\|x\|_p^p}{s}$$

is convex

This function is convex in $s > 0$ and has a non-decreasing derivative such that

$$\frac{\|x\|_p^p}{t^{p-1}}$$

is convex for $p \geq 2$.

Basic Example

Example Problem 2

Show that the function

$$f(x_1, x_2) = -x_1^{\alpha} x_2^{(1-\alpha)}$$

is convex if where $\alpha \in [0, 1]$ and $x_1, x_2 \in \mathbb{R}_{++}$.

Basic Example

Example Problem 2

Show that the function

$$f(x_1, x_2) = -x_1^{\alpha} x_2^{(1-\alpha)}$$

is convex if where $\alpha \in [0, 1]$ and $x_1, x_2 \in \mathbb{R}_{++}$.

Chosen approach: 2nd order condition for convexity.

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$$\nabla f(x_1, x_2) = -\begin{bmatrix} \alpha x_1^{\alpha-1} x_2^{(1-\alpha)} \\ (1-\alpha) x_1^{\alpha} x_2^{-\alpha} \end{bmatrix}$$

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$$\nabla f(x_1, x_2)^2 = -\begin{bmatrix} \alpha(\alpha-1) x_1^{\alpha-2} x_2^{(1-\alpha)} & \alpha(1-\alpha) x_1^{\alpha-1} x_2^{(-\alpha)} \\ \alpha(1-\alpha) x_1^{\alpha-1} x_2^{(-\alpha)} & (1-\alpha)(-\alpha) x_1^{\alpha} x_2^{(-\alpha-1)} \end{bmatrix}$$

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Next Subsection

General Form Problems

General Form Problem

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General Form Problem

Linear Program

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} - d \\ \text{s.t.} \quad & \mathbf{G}\mathbf{x} \preceq \mathbf{h} \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$

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Quadratic Program

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{x}^T\mathbf{Q}\mathbf{x} - \mathbf{q}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{G}\mathbf{x} \preceq \mathbf{h} \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$

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Second Order Cone Program

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \|\mathbf{A}_i\mathbf{x} + \mathbf{b}_i\|_2 \preceq \mathbf{0} \; \forall \; i = 1, \cdots, m \\ & \mathbf{A}_0\mathbf{x} + \mathbf{b}_0 = \mathbf{0} \end{aligned}$$

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Semidefinite Program

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & x_1 F1 + \cdots + x_n F_n + G \preceq \mathbf{0} \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$

Converting Convex Problems

Why bother converting?

- An additional method to show that a problem is convex
- Specific solvers may be designed for certain problem classes i.e LP, QP
- Use of such solvers can result in must computation.

Linear Programs

General Form Problem

Linear Program

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Standard Form Problem

Linear Program

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} - d \\ \text{s.t.} \quad & \mathbf{x} \succeq \mathbf{0} \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$

Example Problem From Text Book

Example Problem 1

Consider the problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{x}\|_1 \\ \text{s.t.} \quad & \|\mathbf{A}^T\mathbf{x} - \mathbf{b}\|_\infty \leq 1 \end{aligned}$$

Convert the problem to a standard form problem of your choice.

Example Problem From Text Book

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$$\min_{\mathbf{x}} \quad \|\mathbf{x}\|_1$$
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Convert the problem to a standard form problem of your choice.

Is this a convex problem?

Example Problem From Text Book

$$\min_{\mathbf{x}} \quad \|\mathbf{x}\|_1 = \sum_{i=1}^{N} |x_i|$$
$$\text{s.t.} \quad \|\mathbf{A}^T\mathbf{x} - \mathbf{b}\|_\infty \leq 1$$

Example Problem From Text Book

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We will address the objective first.

Example Problem From Text Book

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We will address the objective first. We can introduce the additional vector variable **v** such that

$$\begin{aligned} \min_{\mathbf{x}} \quad & \sum_{i=1}^{N} v_i = \mathbf{1}^T\mathbf{v} \\ \text{s.t.} \quad & \|\mathbf{A}^T\mathbf{x} - \mathbf{b}\|_\infty \leq 1 \\ & |x_i| \leq v_i \ \forall\ i = 1, \cdots, N \end{aligned}$$

Example Problem From Text Book

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Example Problem From Text Book

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We can then address the first constraint by noting its equivalence to

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{1}^T \mathbf{v} \\ \text{s.t.} \quad & |\mathbf{a}_i^T \mathbf{x} - b_i| \leq 1 \ \forall \ i = 1, \cdots, N \\ & |x_i| \leq v_i \ \forall \ i = 1, \cdots, N \end{aligned}$$

Example Problem From Text Book

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Finally we can use the fact that all our variables are real valued to rewrite the inequality constraints as affine inequality constraints.

Example Problem From Text Book

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{1}^T \mathbf{v} \\ \text{s.t.} \quad & |\mathbf{a}_i^T \mathbf{x} - b_i| \leq 1 \ \forall\ i = 1, \cdots, N \\ & |x_i| \leq v_i \ \forall\ i = 1, \cdots, N \end{aligned}$$

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$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{1}^T \mathbf{v} \\ \text{s.t.} \quad & -1 \leq \mathbf{a}_i^T \mathbf{x} - b_i \leq 1 \ \forall\ i = 1, \cdots, N \\ & -v_i \leq x_i \leq v_i \ \forall\ i = 1, \cdots, N \end{aligned}$$

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The resulting problem is therefore an LP

Example Problem From Text Book

$$
\begin{aligned}
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\text{s.t.} \quad & -1 \leq \mathbf{a}_i^T \mathbf{x} - b_i \leq 1 \ \forall \ i = 1, \cdots, N \\
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\end{aligned}
$$

Writing this in the general linear program form

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & \mathbf{1}^T \mathbf{v} \\
\text{s.t.} \quad & \begin{bmatrix} \mathbf{a}_1^T & \mathbf{0}^T \\ \vdots & \vdots \\ \mathbf{a}_N^T & \mathbf{0}^T \\ -\mathbf{a}_1^T & \mathbf{0}^T \\ \vdots & \\ -\mathbf{a}_N^T & \mathbf{0}^T \\ \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{v} \end{bmatrix} + \begin{bmatrix} 1 - b_1 \\ \vdots \\ 1 - b_N \\ -1 - b_1 \\ -1 - b_N \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \succeq \mathbf{0}
\end{aligned}
$$

Example Problem From Text Book

Example Problem 2a)

Consider the problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & \frac{\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_1}{\mathbf{c}^T\mathbf{x} + d} \\ \text{s.t} \quad & \|\mathbf{x}\|_\infty \leq 1 \end{aligned}$$

where $d > \|\mathbf{c}\|_1$.

Example Problem From Text Book

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Consider the problem

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where $d > \|\mathbf{c}\|_1$.

Show that the problem is quasi-convex.

Example Problem From Text Book

$$\begin{aligned} \min_{\mathbf{x}} \quad & \frac{\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_1}{\mathbf{c}^T\mathbf{x}+d} \\ \text{s.t} \quad & \|\mathbf{x}\|_\infty \leq 1 \end{aligned}$$

Example Problem From Text Book

$$\begin{aligned} \min_{\mathbf{x}} \quad & \frac{\|\mathbf{Ax} - \mathbf{b}\|_1}{\mathbf{c}^T\mathbf{x} + d} \\ \text{s.t} \quad & \|\mathbf{x}\|_\infty \leq 1 \end{aligned}$$

A function is quasi-convex if and only if all its sublevel sets are convex

Example Problem From Text Book

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A function is quasi-convex if and only if all its sublevel sets are convex i.e.

$$\mathcal{S}_\alpha = \left\{ \mathbf{x} \mid \frac{\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_1}{\mathbf{c}^T\mathbf{x} + d} \leq \alpha \right\}.$$

Example Problem From Text Book

$$\begin{aligned} \min_{\mathbf{x}} \quad & \frac{\|\mathbf{Ax} - \mathbf{b}\|_1}{\mathbf{c}^T\mathbf{x} + d} \\ \text{s.t} \quad & \|\mathbf{x}\|_\infty \leq 1 \end{aligned}$$

A function is quasi-convex if and only if all its sublevel sets are convex i.e.

$$\mathcal{S}_\alpha = \left\{\mathbf{x} \mid \frac{\|\mathbf{Ax} - \mathbf{b}\|_1}{\mathbf{c}^T\mathbf{x} + d} \leq \alpha\right\}.$$

Note that this set can be rephrased as

$$\mathcal{S}_\alpha = \left\{\mathbf{x} \mid \|\mathbf{Ax} - \mathbf{b}\|_1 \leq \alpha\left(\mathbf{c}^T\mathbf{x} + d\right)\right\}$$

Example Problem From Text Book

Plotting the Sub-level Sets in Matlab

Example Problem From Text Book

The constraints

$$\|\mathbf{Ax} - \mathbf{b}\|_1 \leq \alpha \left(\mathbf{c}^T\mathbf{x} + d\right)$$

can be rewritten by introducing the additional vector variable $\mathbf{v}$

Example Problem From Text Book

The constraints

$$\|\mathbf{Ax} - \mathbf{b}\|_1 \leq \alpha \left(\mathbf{c}^T \mathbf{x} + d\right)$$

can be rewritten by introducing the additional vector variable **v** such that

$$\mathbf{1}^T \mathbf{v} \leq \alpha \left(\mathbf{c}^T \mathbf{x} + d\right)$$
$$-\mathbf{v} \leq \mathbf{Ax} - \mathbf{b} \leq \mathbf{v}$$

Example Problem From Text Book

The constraints

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i.e. as a set of standard affine inequalities and thus can be interpreted as an intersection of half-spaces.

Example Problem From Text Book

Example 2b)

Show that the problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & \frac{\|\mathbf{Ax} - \mathbf{b}\|_1}{\mathbf{c}^T\mathbf{x} + d} \\ \text{s.t} \quad & \|\mathbf{x}\|_\infty \leq 1 \end{aligned}$$

is equivalent to

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{Ay} - \mathbf{b}t\|_1 \\ \text{s.t} \quad & \|\mathbf{y}\|_\infty \leq t \\ & \mathbf{c}^T\mathbf{y} + dt = 1 \end{aligned}$$

and ultimately is equivalent to a linear problem.

Example Problem From Text Book

$$\begin{aligned} \min_{\mathbf{x}} \quad & \frac{\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_1}{\mathbf{c}^T\mathbf{x} + d} \\ \text{s.t} \quad & \|\mathbf{x}\|_\infty \leq 1 \end{aligned}$$

Example Problem From Text Book

$$\begin{aligned} \min_{\mathbf{x}} \quad & \frac{\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_1}{\mathbf{c}^T\mathbf{x}+d} \\ \text{s.t} \quad & \|\mathbf{x}\|_\infty \leq 1 \end{aligned}$$

We begin by introducing the additional variable t such that

$$\begin{aligned} \min_{\mathbf{x}} \quad & t\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_1 \\ \text{s.t} \quad & \|\mathbf{x}\|_\infty \leq 1 \\ & \mathbf{c}^T\mathbf{x}+d \geq \frac{1}{t} \end{aligned}$$

Example Problem From Text Book

$$\begin{aligned} \min_{\mathbf{x}} \quad & \frac{\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_1}{\mathbf{c}^T\mathbf{x}+d} \\ \text{s.t} \quad & \|\mathbf{x}\|_\infty \leq 1 \end{aligned}$$

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Defining $\mathbf{y} = \mathbf{x}t$, it follows that

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{A}\mathbf{y}-\mathbf{b}t\|_1 \\ \text{s.t} \quad & \|\mathbf{y}\|_\infty \leq t \\ & \mathbf{c}^T\mathbf{y}+dt \geq 1 \end{aligned}$$

Example Problem From Text Book

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{A}\mathbf{y} - \mathbf{b}t\|_1 \\ \text{s.t} \quad & \|\mathbf{y}\|_\infty \leq t \\ & \mathbf{c}^T\mathbf{y} + dt \geq 1 \end{aligned}$$

Example Problem From Text Book

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & \|\mathbf{A}\mathbf{y} - \mathbf{b}t\|_1 \\
\text{s.t} \quad & \|\mathbf{y}\|_\infty \leq t \\
& \mathbf{c}^T\mathbf{y} + dt \geq 1
\end{aligned}
$$

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Similarly to Example 1, we introduce the vector variable **v** such that

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{1}^T\mathbf{v} \\ \text{s.t} \quad & \|\mathbf{y}\|_\infty \leq t \\ & \mathbf{c}^T\mathbf{y} + dt \geq 1 \\ & |\mathbf{a}_i\mathbf{y} - \mathbf{b}_i t| \leq v_i \ \forall \ i = 1, \cdots, N \end{aligned}$$

Example Problem From Text Book

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\min_{\mathbf{x}} \quad & \mathbf{1}^T \mathbf{v} \\
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Finally we rewrite the constraints such that

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{1}^T \mathbf{v} \\ \text{s.t} \quad & -t \leq \mathbf{y}_i \leq t \ \forall \ i = 1, \cdots, N \\ & \mathbf{c}^T \mathbf{y} + dt \geq 1 \\ & -v_i \leq \mathbf{a}_i \mathbf{y} - \mathbf{b}_i t \leq v_i \ \forall \ i = 1, \cdots, N \end{aligned}$$

which is in the general form of a linear program

Next Subsection

Lagrangian Duality

Why Use Lagrangian Duality?

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- Provide a convex lower bound of optimal value of primal problem

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- ...

Example Problem From Text Book

Example Problem 1

Consider the general form LP

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} - d \\ \text{s.t.} \quad & \mathbf{G}\mathbf{x} \preceq \mathbf{h} \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$

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Find the its equivalent dual problem.

Example Problem From Text Book

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The Lagrangian of this LP is given by

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\nu}, \boldsymbol{\mu}) = \mathbf{c}^T\mathbf{x} - d + \boldsymbol{\nu}^T(\mathbf{G}\mathbf{x} - \mathbf{h}) + \boldsymbol{\mu}^T(\mathbf{A}\mathbf{x} - \mathbf{b}), \ \text{s.t.} \ \boldsymbol{\nu} \geq \mathbf{0}.$$

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As this is an affine function of $\mathbf{x}$ the dual function is given by

Example Problem From Text Book

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$$g(\boldsymbol{\nu}, \boldsymbol{\mu}) = \inf_{\mathbf{x}} \left(\mathbf{c}^T\mathbf{x} - d + \boldsymbol{\nu}^T(\mathbf{G}\mathbf{x} - \mathbf{h}) + \boldsymbol{\mu}^T(\mathbf{A}\mathbf{x} - \mathbf{b})\right)$$

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As this is an affine function of $\mathbf{x}$ the dual function is given by

$$\begin{aligned} g(\boldsymbol{\nu}, \boldsymbol{\mu}) &= \inf_{\mathbf{x}} \left(\mathbf{c}^T\mathbf{x} - d + \boldsymbol{\nu}^T(\mathbf{G}\mathbf{x} - \mathbf{h}) + \boldsymbol{\mu}^T(\mathbf{A}\mathbf{x} - \mathbf{b}) \right) \\ &= \begin{cases} -\boldsymbol{\nu}^T\mathbf{h} - \boldsymbol{\mu}^T\mathbf{b} - d & \text{if } \mathbf{c} + \mathbf{G}^T\boldsymbol{\nu} + \mathbf{A}^T\boldsymbol{\mu} = \mathbf{0} \\ -\infty & \text{otherwise} \end{cases}. \end{aligned}$$

Example Problem From Text Book

The dual problem is therefore

$$\begin{aligned} \min_{\mathbf{x}} \quad & -g(\boldsymbol{\nu}, \boldsymbol{\mu}) \\ \text{s.t.} \quad & \boldsymbol{\nu} \geq \mathbf{0} \end{aligned}$$

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Therefore substituting the definition of g

$$\begin{aligned} \min_{\mathbf{x}} \quad & \boldsymbol{\nu}^T \mathbf{h} + \boldsymbol{\mu}^T \mathbf{b} + d \\ \text{s.t.} \quad & \boldsymbol{\nu} \geq \mathbf{0} \\ & \mathbf{c} + \mathbf{G}^T \boldsymbol{\nu} + \mathbf{A}^T \boldsymbol{\mu} = \mathbf{0} \end{aligned}$$

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Which is interestingly another LP in standard form.

Example Problem From Text Book

Example Problem 2

Consider the linearly constrained norm problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{x}\|_1 \\ \text{s.t.} \quad & \mathbf{C}\mathbf{x} = \mathbf{d} \end{aligned}$$

Example Problem From Text Book

Example Problem 2

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Example Problem From Text Book

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The dual function is therefore given by

$$\begin{aligned} g(\boldsymbol{\lambda}) &= \inf_{\mathbf{x}} \left(\|\mathbf{x}\|_1 - \boldsymbol{\lambda}^T (\mathbf{C}\mathbf{x} - \mathbf{d}) \right) \\ &= -\sup_{\mathbf{x}} \left(\boldsymbol{\lambda}^T (\mathbf{C}\mathbf{x} - \mathbf{d}) - \|\mathbf{x}\|_1 \right) \\ &= -f^*(\boldsymbol{\lambda}) \end{aligned}$$

Example Problem From Text Book

$$g(\boldsymbol{\lambda}) = -\sup_{\mathbf{x}} \left(\boldsymbol{\lambda}^T (\mathbf{C}\mathbf{x} - \mathbf{d}) - \|\mathbf{x}\|_1 \right)$$

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$$

Note that

$$
\mathbf{u}^T \mathbf{x} \leq \|\mathbf{x}\|_1 \iff \sup \left\{ \left(\mathbf{u}^T \mathbf{x} \right) \mid \|\mathbf{x}\|_1 \leq 1 \right\} \leq 1
$$

Example Problem From Text Book

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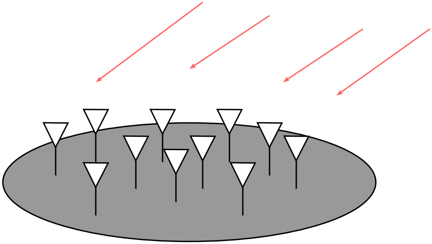
The final dual problem is given by

$$\begin{aligned} \min_{\boldsymbol{\lambda}} \quad & \boldsymbol{\lambda}^T \mathbf{d} \\ \textbf{s.t.} \quad & \|\mathbf{C}^T \boldsymbol{\lambda}\|_\infty \leq 1 \end{aligned}$$

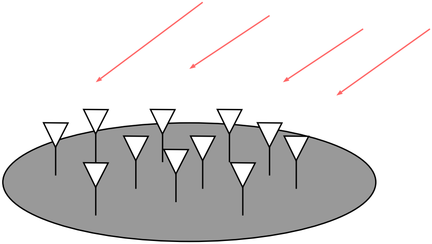
Example Problem: Channel Capacity Maximisation



Channel Capacity Maximisation

Given an N element transmit antenna array, how can we distribute transmission power to maximise the rate of data transmission back to a target source?

Example Problem: Channel Capacity Maximisation



Channel Capacity Maximisation

Given an N element transmit antenna array, how can we distribute transmission power to maximise the rate of data transmission back to a target source?

In the case of an additive Gaussian channel, this corresponds to

$$\begin{aligned} \max_{\mathbf{x}} \quad & \sum_{i=1}^{N} \log_2 \left(1 + \frac{x_i}{\sigma_i}\right) \\ \text{s.t.} \quad & x_i \geq 0 \; \forall \; i = 1, \cdots, N \\ & \mathbf{1}^T \mathbf{x} = 1 \end{aligned}$$

where σ_i is the bandwidth and noise variance per antenna.

Example Problem: Channel Capacity Maximisation

$$\begin{aligned} \max_{\mathbf{x}} \quad & \sum_{i=1}^{N} B_i \log_2\left(1 + \frac{x_i}{\sigma_i}\right) \\ \text{s.t.} \quad & x_i \geq 0 \ \forall \ i = 1, \cdots, N \\ & \mathbf{1}^T \mathbf{x} = 1 \end{aligned}$$

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Is the problem convex?

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Objective
As the composition of a non-decreasing concave function and an affine function, the objective is concave. Maximising a concave function is equivalent to minimising a convex function.

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As the composition of a non-decreasing concave function and an affine function, the objective is concave. Maximising a concave function is equivalent to minimising a convex function.

Domain
The constraints are affine by nature therefore define a convex set.

Channel Capacity Maximisation

Using log laws the objective can be rewritten as

$$\sum_{i=1}^{N} \log_2 \left(1 + \frac{x_i}{\sigma_i}\right) = \frac{1}{\log(2)} \sum_{i=1}^{N} \left(\log\left(\sigma_i + x_i\right) - \log\left(\sigma_i\right)\right)$$

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How can we go about solving this?

Channel Capacity Maximisation

The Lagrangian of this problem is given by

$$\mathcal{L}(x, \boldsymbol{\nu}, \mu) = \sum_{i=1}^{N} \left(-\log\left(\sigma_i + x_i\right) + \nu_i x_i - \mu x_i\right) - \mu$$

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$$\text{Primal Optimality} \qquad -\frac{1}{\sigma_i + x_i^*} - \nu_i^* + \mu^* = 0$$

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\text{Dual Feasibility} \quad & \nu_i^* \geq 0 \; \forall \; i = 1, \cdots, N \\
\text{Complementary Slackness} \quad & \nu_i^* x_i^* = 0 \; \forall \; i = 1, \cdots, N
\end{aligned}$$

Channel Capacity Maximisation

Primal optimality implies that

$$\nu_i^* = \mu^* - \frac{1}{\sigma_i + x_i^*}$$

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$$x_i^* = \begin{cases} 0 & \text{if } \mu^* > \frac{1}{\sigma_i} \\ \frac{1}{\mu^*} - \sigma_i & \text{otherwise} \end{cases}$$

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If $\mu^* < \frac{1}{\sigma_i}$ this can only hold if $x_i^* > 0$ in which case complementary slackness means $\nu_i^* = 0$.

$$x_i^* = \begin{cases} 0 & \text{if } \mu^* > \frac{1}{\sigma_i} \\ \frac{1}{\mu^*} - \sigma_i & \text{otherwise} \end{cases}$$

Finally, primal feasibility implies that

$$\sum_{i \in C} \left(\frac{1}{\mu^*} - \sigma_i \right) = 1$$

where C is the set of i's such that $\mu^* > \frac{1}{\sigma}$.

Channel Capacity Maximisation

Primal optimality implies that

$$\nu_i^* = \mu^* - \frac{1}{\sigma_i + x_i^*}$$

which, combined with dual feasibility implies

$$\frac{1}{\sigma_i + x_i^*} \leq \mu^*$$

If $\mu^* < \frac{1}{\sigma_i}$ this can only hold if $x_i^* > 0$ in which case complementary slackness means $\nu_i^* = 0$.

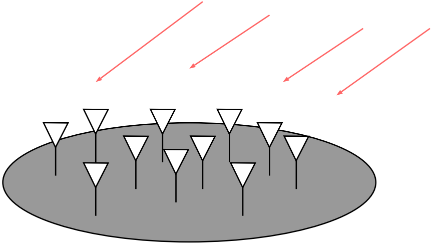
$$x_i^* = \begin{cases} 0 & \text{if } \mu^* > \frac{1}{\sigma_i} \\ \frac{1}{\mu^*} - \sigma_i & \text{otherwise} \end{cases}$$

Finally, primal feasibility implies that

$$\sum_{i \in C} \left(\frac{1}{\mu^*} - \sigma_i \right) = 1$$

where C is the set of i's such that $\mu^* > \frac{1}{\sigma}$. Solve with bisection method to find μ^* and thus $\mathbf{x}^*$.

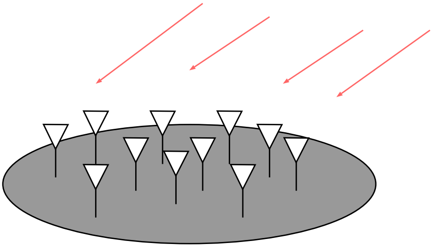
Channel Capacity Maximisation



Alternatively Solve with CVX

```
cvx_begin
  variable x(n)
  minimize sum(-log(sigma + x)))
  subject to
    ones(n,1)' * x == 1
    x >= 0
cvx_end
```

Channel Capacity Maximisation



Dual Variables with CVX

```
cvx_begin
    variable x(n)
    dual variable mu
    dual variable nu(n)
    minimize sum(-log(sigma + x)))
    subject to
        mu: ones(n,1)' * x == 1
        nu: x >= 0
cvx_end
```

Channel Capacity Maximisation

Demo in Matlab