ET4350
Applied Convex Optimization
Lecture 11

ℓ_1-norm heuristics for cardinality problems

- cardinality problems arise often, but are hard to solve exactly
- a simple heuristic, that relies on ℓ_1-norm, seems to work well
- used for many years, in many fields
  - sparse design
  - LASSO, robust estimation in statistics
  - support vector machine (SVM) in machine learning
  - total variation reconstruction in signal processing, geophysics
  - compressed sensing
- new theoretical results guarantee the method works, at least for a few problems

Cardinality

- the **cardinality** of $x \in \mathbf{R}^n$, denoted $\mathbf{card}(x)$, is the number of nonzero components of x
- $\mathbf{card}$ is separable; for scalar x, $\mathbf{card}(x) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}$
- $\mathbf{card}$ is quasiconcave on $\mathbf{R}^n_+$ (but not $\mathbf{R}^n$) since

$$\mathbf{card}(x + y) \geq \min\{\mathbf{card}(x), \mathbf{card}(y)\}$$

  holds for $x, y \succeq 0$
- but otherwise has no convexity properties
- arises in many problems

General convex-cardinality problems

a **convex-cardinality problem** is one that would be convex, except for appearance of $\mathbf{card}$ in objective or constraints

examples (with $\mathcal{C}$, f convex):

- convex minimum cardinality problem:

$$\begin{array}{ll} \text{minimize} & \mathbf{card}(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

- convex problem with cardinality constraint:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C}, \quad \mathbf{card}(x) \leq k \end{array}$$

Solving convex-cardinality problems

convex-cardinality problem with $x \in \mathbf{R}^n$

- if we fix the sparsity pattern of x (*i.e.*, which entries are zero/nonzero) we get a convex problem
- by solving 2^n convex problems associated with all possible sparsity patterns, we can solve convex-cardinality problem
(possibly practical for $n \leq 10$; not practical for $n > 15$ or so . . .)
- general convex-cardinality problem is (NP-) hard
- can solve globally by branch-and-bound
  - can work for particular problem instances (with some luck)
  - in worst case reduces to checking all (or many of) 2^n sparsity patterns

Boolean LP as convex-cardinality problem

- Boolean LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b, \quad x_i \in \{0,1\} \end{array}$$

  includes many famous (hard) problems, *e.g.*, 3-SAT, traveling salesman

- can be expressed as

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b, \quad \mathbf{card}(x) + \mathbf{card}(1-x) \leq n \end{array}$$

  since $\mathbf{card}(x) + \mathbf{card}(1-x) \leq n \iff x_i \in \{0,1\}$

- conclusion: general convex-cardinality problem is hard

Sparse design

$$\begin{array}{ll} \text{minimize} & \mathbf{card}(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

- find sparsest design vector x that satisfies a set of specifications
- zero values of x simplify design, or correspond to components that aren't even needed
- examples:
  - FIR filter design (zero coefficients reduce required hardware)
  - antenna array beamforming (zero coefficients correspond to unneeded antenna elements)
  - truss design (zero coefficients correspond to bars that are not needed)
  - wire sizing (zero coefficients correspond to wires that are not needed)

Sparse modeling / regressor selection

fit vector $b \in \mathbf{R}^m$ as a linear combination of k regressors (chosen from n possible regressors)

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2 \\ \text{subject to} & \mathbf{card}(x) \leq k \end{array}$$

- gives k-term model
- chooses subset of k regressors that (together) best fit or explain b
- can solve (in principle) by trying all $\binom{n}{k}$ choices
- variations:
  - minimize $\mathbf{card}(x)$ subject to $\|Ax - b\|_2 \leq \epsilon$
  - minimize $\|Ax - b\|_2 + \lambda \, \mathbf{card}(x)$

Sparse signal reconstruction

- estimate signal x, given
  - noisy measurement $y = Ax + v$, $v \sim \mathcal{N}(0, \sigma^2 I)$ (A is known; v is not)
  - prior information $\mathbf{card}(x) \leq k$

- maximum likelihood estimate $\hat{x}_{\mathrm{ml}}$ is solution of

$$\begin{array}{ll} \text{minimize} & \|Ax - y\|_2 \\ \text{subject to} & \mathbf{card}(x) \leq k \end{array}$$

Estimation with outliers

- we have measurements $y_i = a_i^T x + v_i + w_i$, $i = 1, \ldots, m$
- noises $v_i \sim \mathcal{N}(0, \sigma^2)$ are independent
- only assumption on w is sparsity: $\mathbf{card}(w) \leq k$
- $\mathcal{B} = \{i \mid w_i \neq 0\}$ is set of bad measurements or *outliers*
- maximum likelihood estimate of x found by solving

$$\begin{array}{ll} \text{minimize} & \sum_{i \notin \mathcal{B}} (y_i - a_i^T x)^2 \\ \text{subject to} & |\mathcal{B}| \leq k \end{array}$$

  with variables x and $\mathcal{B} \subseteq \{1, \ldots, m\}$

- equivalent to

$$\begin{array}{ll} \text{minimize} & \|y - Ax - w\|_2^2 \\ \text{subject to} & \mathbf{card}(w) \leq k \end{array}$$

Minimum number of violations

- set of convex inequalities

$$f_1(x) \leq 0, \ \ldots, \ f_m(x) \leq 0, \qquad x \in \mathcal{C}$$

- choose x to minimize the number of violated inequalities:

$$\begin{array}{ll} \text{minimize} & \mathbf{card}(t) \\ \text{subject to} & f_i(x) \leq t_i, \quad i = 1, \ldots, m \\ & x \in \mathcal{C}, \quad t \geq 0 \end{array}$$

- determining whether zero inequalities can be violated is (easy) convex feasibility problem

Portfolio investment with linear and fixed costs

- we use budget B to purchase (dollar) amount $x_i \geq 0$ of stock i
- trading fee is fixed cost plus linear cost: $\beta\,\mathbf{card}(x) + \alpha^T x$
- budget constraint is $\mathbf{1}^T x + \beta\,\mathbf{card}(x) + \alpha^T x \leq B$
- mean return on investment is $\mu^T x$; variance is $x^T \Sigma x$
- minimize investment variance (risk) with mean return $\geq R_{\min}$:

$$
\begin{array}{ll}
\text{minimize} & x^T \Sigma x \\
\text{subject to} & \mu^T x \geq R_{\min}, \quad x \succeq 0 \\
& \mathbf{1}^T x + \beta\,\mathbf{card}(x) + \alpha^T x \leq B
\end{array}
$$

ℓ_1-norm heuristic

- replace $\mathbf{card}(z)$ with $\gamma\|z\|_1$, or add regularization term $\gamma\|z\|_1$ to objective

- $\gamma > 0$ is parameter used to achieve desired sparsity (when $\mathbf{card}$ appears in constraint, or as term in objective)

- more sophisticated versions use $\sum_i w_i|z_i|$ or $\sum_i w_i(z_i)_+ + \sum_i v_i(z_i)_-$, where w, v are positive weights

Example: Minimum cardinality problem

- start with (hard) minimum cardinality problem

$$\begin{array}{ll} \text{minimize} & \mathbf{card}(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

($\mathcal{C}$ convex)

- apply heuristic to get (easy) ℓ_1-norm minimization problem

$$\begin{array}{ll} \text{minimize} & \|x\|_1 \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

Example: Cardinality constrained problem

- start with (hard) cardinality constrained problem (f, $\mathcal{C}$ convex)

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C}, \quad \mathbf{card}(x) \leq k \end{array}$$

- apply heuristic to get (easy) ℓ_1-constrained problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathcal{C}, \quad \|x\|_1 \leq \beta \end{array}$$

  or ℓ_1-regularized problem

$$\begin{array}{ll} \text{minimize} & f(x) + \gamma\|x\|_1 \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

  β, γ adjusted so that $\mathbf{card}(x) \leq k$

Interpretation as convex relaxation

- start with

$$\begin{array}{ll} \text{minimize} & \mathbf{card}(x) \\ \text{subject to} & x \in \mathcal{C}, \quad \|x\|_\infty \leq R \end{array}$$

- equivalent to mixed Boolean convex problem

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T z \\ \text{subject to} & |x_i| \leq R z_i, \quad i = 1, \dots, n \\ & x \in \mathcal{C}, \quad z_i \in \{0, 1\}, \quad i = 1, \dots, n \end{array}$$

with variables x, z

- now relax $z_i \in \{0,1\}$ to $z_i \in [0,1]$ to obtain

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T z \\ \text{subject to} & |x_i| \leq R z_i, \quad i = 1, \ldots, n \\ & x \in \mathcal{C} \\ & 0 \leq z_i \leq 1, \quad i = 1, \ldots, n \end{array}$$

which is equivalent to

$$\begin{array}{ll} \text{minimize} & (1/R)\|x\|_1 \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

the ℓ_1 heuristic

- optimal value of this problem is lower bound on original problem

Sparse signal reconstruction

- convex-cardinality problem:

$$\begin{array}{ll} \text{minimize} & \|Ax - y\|_2 \\ \text{subject to} & \mathbf{card}(x) \leq k \end{array}$$

- ℓ_1 heuristic:

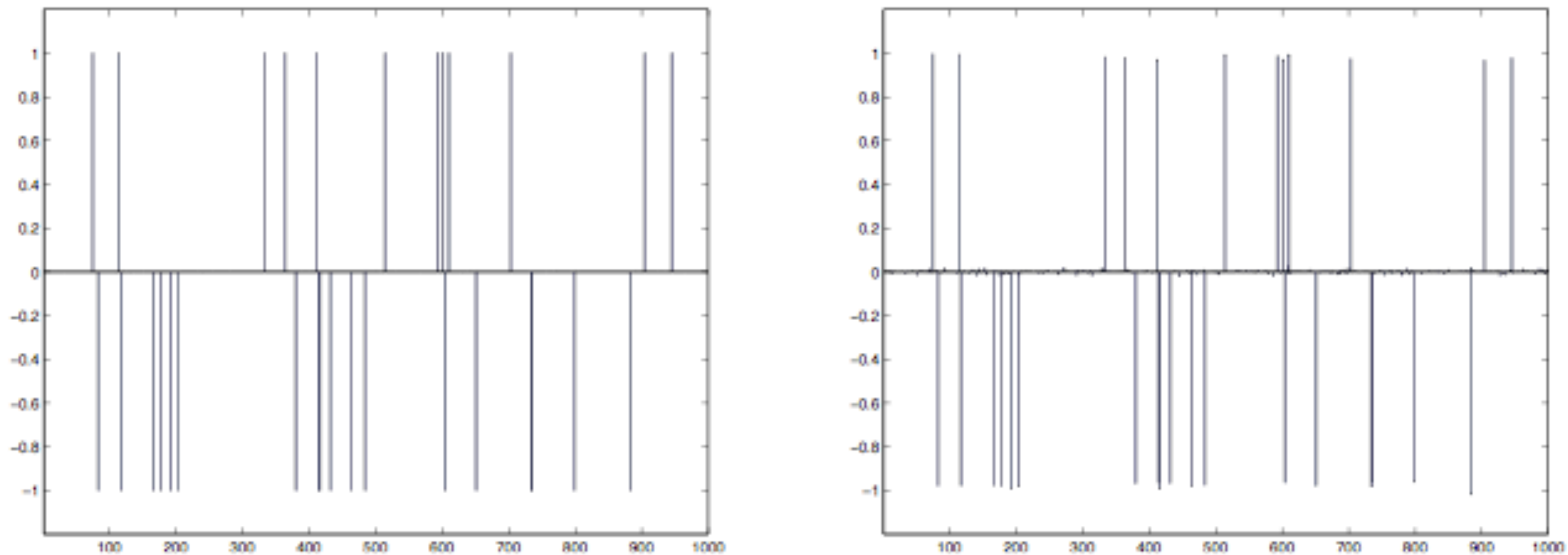
$$\begin{array}{ll} \text{minimize} & \|Ax - y\|_2 \\ \text{subject to} & \|x\|_1 \leq \beta \end{array}$$

  (called LASSO)

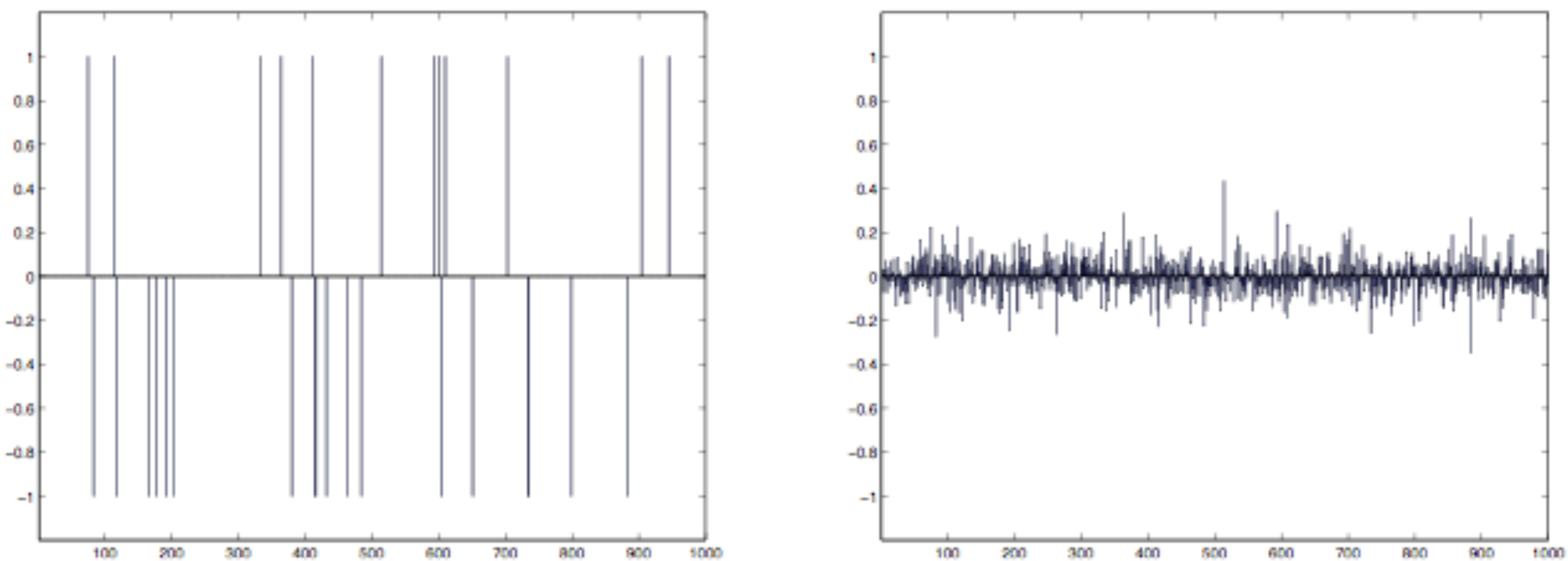
- another form: minimize $\|Ax - y\|_2 + \gamma\|x\|_1$
  (called basis pursuit denoising)

Example

- signal $x \in \mathbf{R}^n$ with $n = 1000$, $\mathbf{card}(x) = 30$
- $m = 200$ (random) noisy measurements: $y = Ax + v$, $v \sim \mathcal{N}(0, \sigma^2\mathbf{1})$, $A_{ij} \sim \mathcal{N}(0, 1)$
- *left*: original; *right*: ℓ_1 reconstruction with $\gamma = 10^{-3}$



- ℓ_2 reconstruction; minimizes $\|Ax - y\|_2 + \gamma\|x\|_2$, where $\gamma = 10^{-3}$
- *left*: original; *right*: ℓ_2 reconstruction



Some recent theoretical results

- suppose $y = Ax$, $A \in \mathbf{R}^{m \times n}$, $\mathbf{card}(x) \leq k$
- to reconstruct x, clearly need $m \geq k$
- if $m \geq n$ and A is full rank, we can reconstruct x without cardinality assumption
- when does the ℓ_1 heuristic (minimizing $\|x\|_1$ subject to $Ax = y$) reconstruct x (exactly)?

recent results by Candès, Donoho, Romberg, Tao, . . .

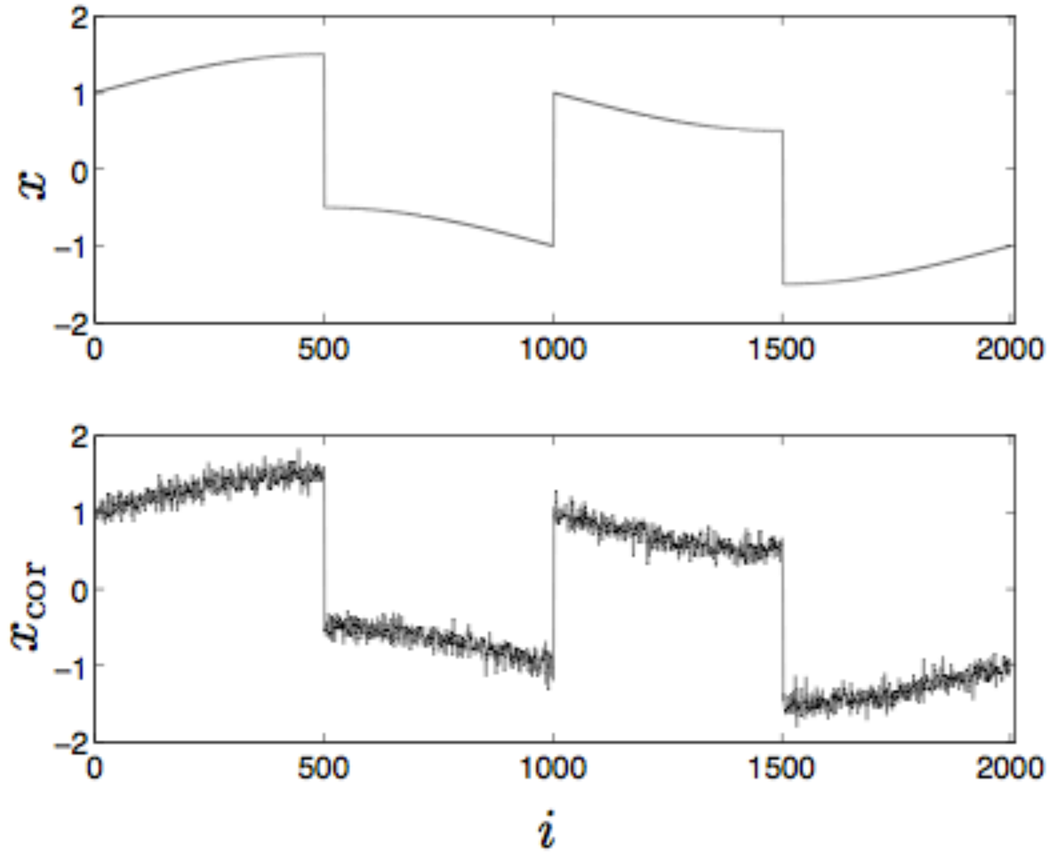
- (for some choices of A) if $m \geq (C \log n)k$, ℓ_1 heuristic reconstructs x exactly, with overwhelming probability
- C is absolute constant; valid A's include
  - $A_{ij} \sim \mathcal{N}(0, \sigma^2)$
  - Ax gives Fourier transform of x at m frequencies, chosen from uniform distribution

Total variation reconstruction

- fit x_{cor} with piecewise constant $\hat{x}$, no more than k jumps
- convex-cardinality problem: minimize $\|\hat{x} - x_{\text{cor}}\|_2$ subject to $\mathbf{card}(Dx) \leq k$ (D is first order difference matrix)
- heuristic: minimize $\|\hat{x} - x_{\text{cor}}\|_2 + \gamma\|Dx\|_1$; vary γ to adjust number of jumps
- $\|Dx\|_1$ is *total variation* of signal $\hat{x}$
- method is called *total variation reconstruction*
- unlike ℓ_2 based reconstruction, TVR filters high frequency noise out while preserving sharp jumps

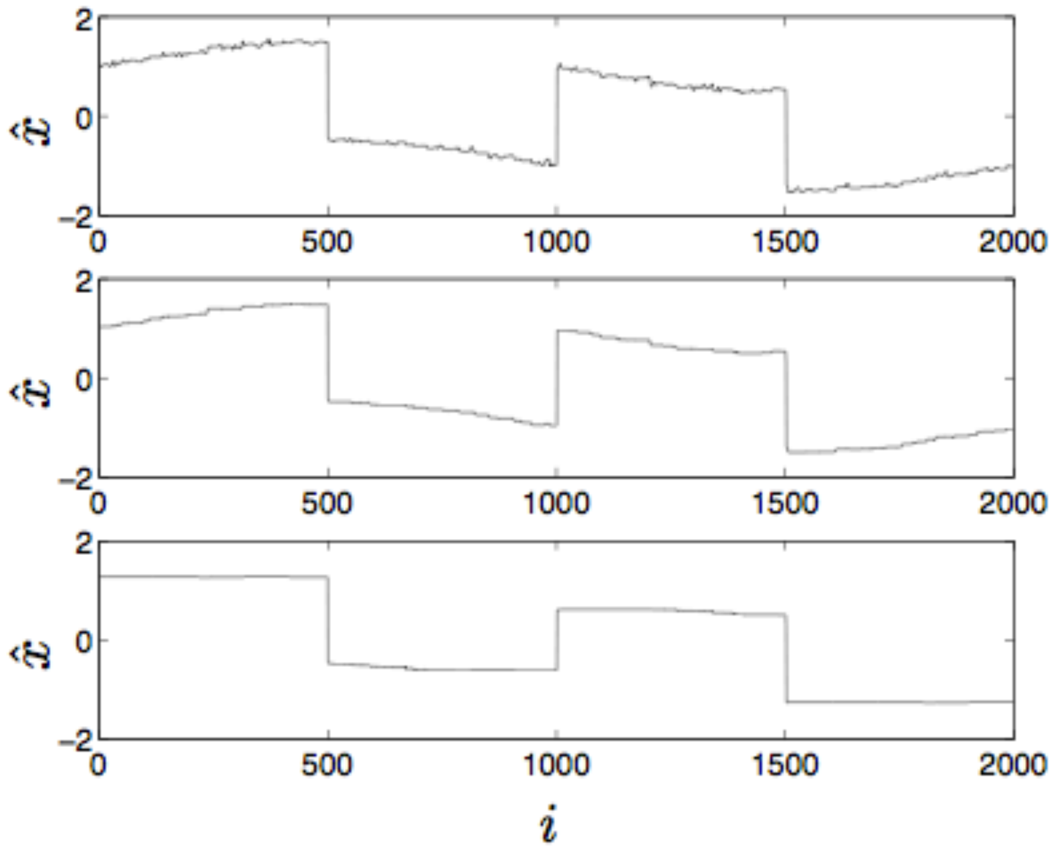
Example (§6.3.3 in BV book)

signal $x \in \mathbf{R}^{2000}$ and corrupted signal $x_{\mathrm{cor}} \in \mathbf{R}^{2000}$



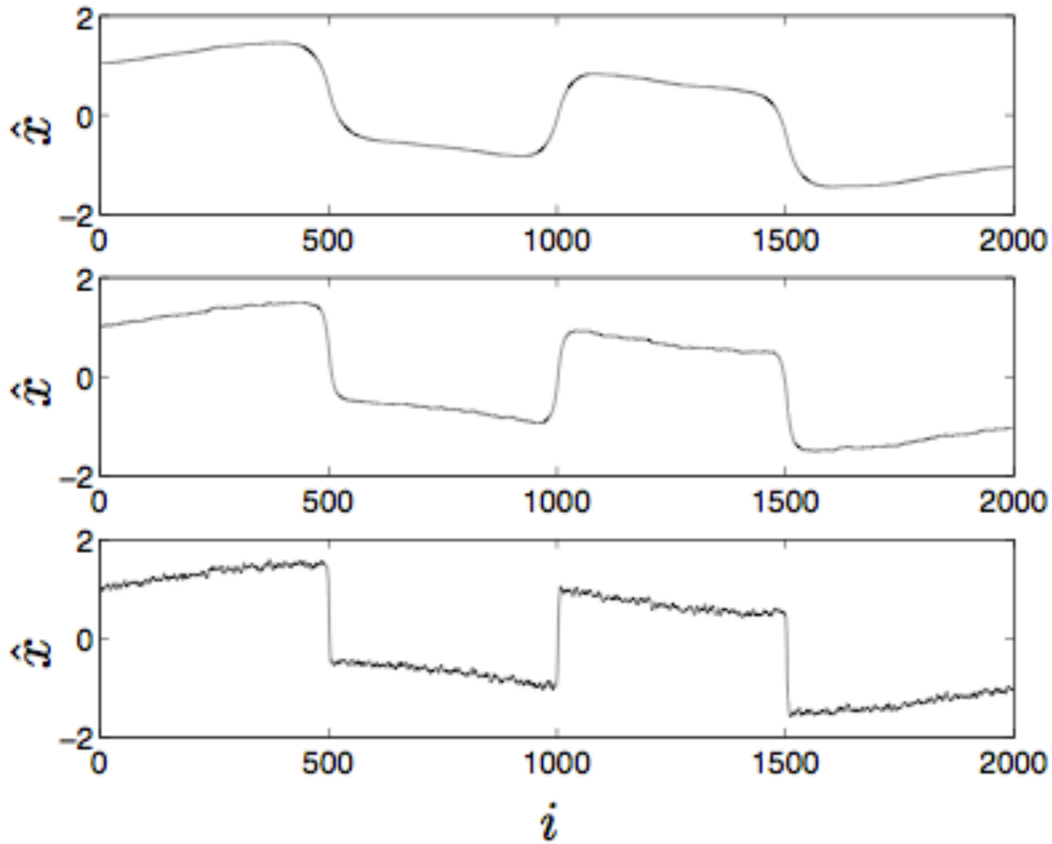
Total variation reconstruction

for three values of γ



ℓ_2 reconstruction

for three values of γ



Example: 2D total variation reconstruction

- $x \in \mathbf{R}^n$ are values of pixels on $N \times N$ grid ($N = 31$, so $n = 961$)
- assumption: x has relatively few big changes in value (*i.e.*, boundaries)
- we have $m = 120$ linear measurements, $y = Fx$ ($F_{ij} \sim \mathcal{N}(0,1)$)
- as convex-cardinality problem:

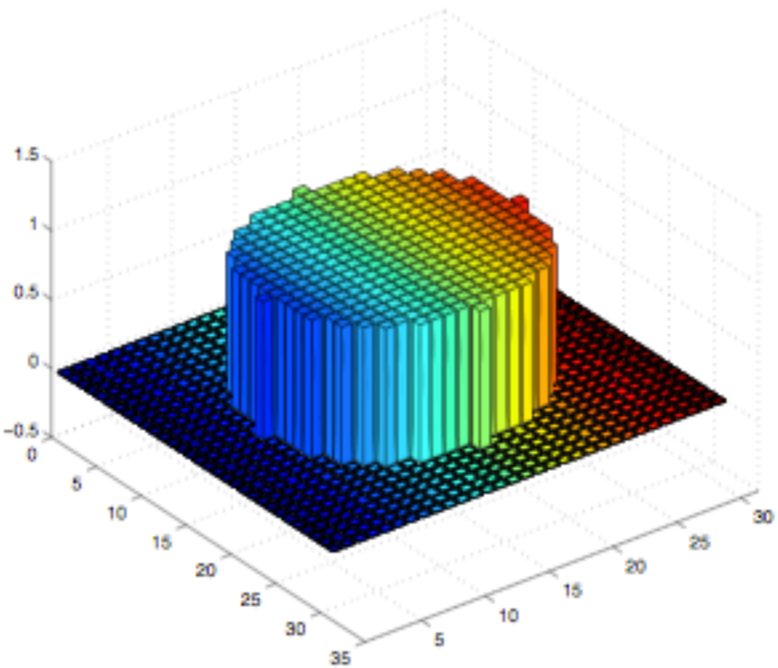
$$\begin{array}{ll} \text{minimize} & \mathbf{card}(x_{i,j} - x_{i+1,j}) + \mathbf{card}(x_{i,j} - x_{i,j+1}) \\ \text{subject to} & y = Fx \end{array}$$

- ℓ_1 heuristic (objective is a 2D version of total variation)

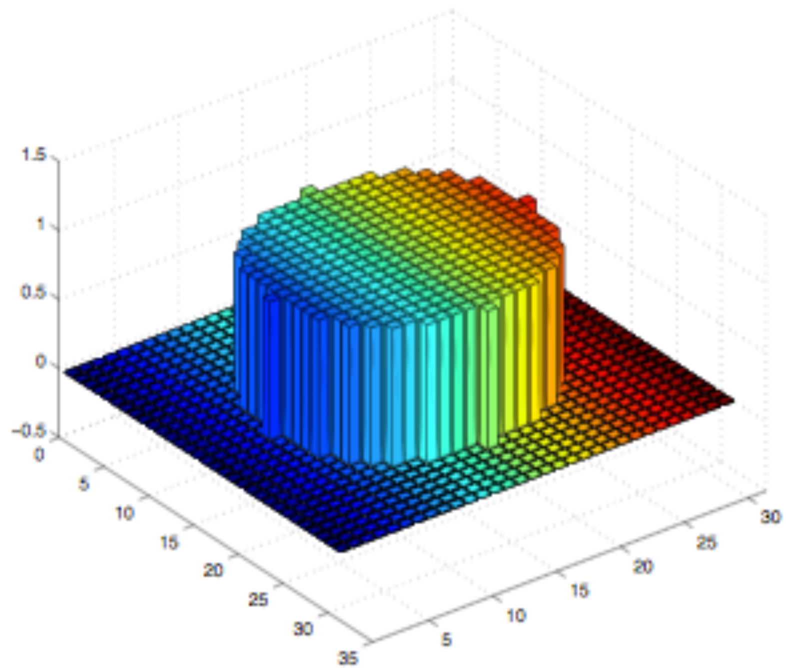
$$\begin{array}{ll} \text{minimize} & \sum |x_{i,j} - x_{i+1,j}| + \sum |x_{i,j} - x_{i,j+1}| \\ \text{subject to} & y = Fx \end{array}$$

TV reconstruction

original

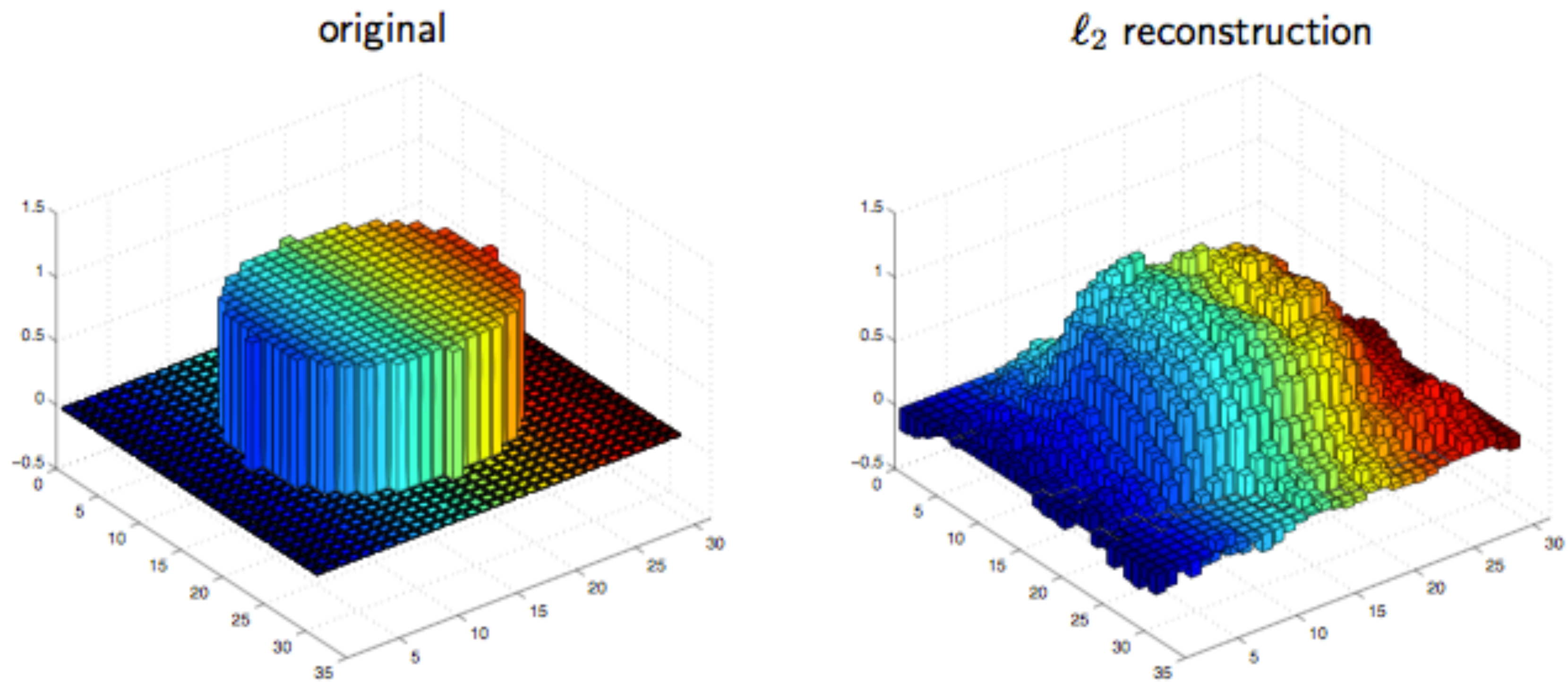


TV reconstruction



. . . not bad for $8\times$ more variables than measurements!

ℓ_2 reconstruction



. . . this is what you'd expect with $8\times$ more variables than measurements

Extension to matrices

- $\mathbf{Rank}$ is natural analog of $\mathbf{card}$ for matrices

- convex-rank problem: convex, except for $\mathbf{Rank}$ in objective or constraints

- rank problem reduces to card problem when matrices are diagonal: $\mathbf{Rank}(\mathbf{diag}(x)) = \mathbf{card}(x)$

- analog of ℓ_1 heuristic: use *nuclear norm*, $\|X\|_* = \sum_i \sigma_i(X)$ (sum of singular values; dual of spectral norm)

- for $X \succeq 0$, reduces to $\mathbf{Tr}\, X$ (for $x \succeq 0$, $\|x\|_1$ reduces to $\mathbf{1}^T x$)

Factor modeling

- given matrix $\Sigma \in \mathbf{S}_+^n$, find approximation of form $\hat{\Sigma} = FF^T + D$, where $F \in \mathbf{R}^{n \times r}$, D is diagonal nonnegative
- gives underlying factor model (with r factors)

$$x = Fz + v, \quad v \sim \mathcal{N}(0, D), \quad z \sim \mathcal{N}(0, I)$$

- model with fewest factors:

$$\begin{array}{ll} \text{minimize} & \textbf{Rank}\, X \\ \text{subject to} & X \succeq 0, \quad D \succeq 0 \text{ diagonal} \\ & X + D \in \mathcal{C} \end{array}$$

with variables $D,\ X \in \mathbf{S}^n$
$\mathcal{C}$ is convex set of acceptable approximations to Σ

Example

- $x = Fz + v$, $z \sim \mathcal{N}(0, I)$, $v \sim \mathcal{N}(0, D)$, D diagonal; $F \in \mathbf{R}^{20 \times 3}$
- Σ is empirical covariance matrix from $N = 3000$ samples
- set of acceptable approximations

$$\mathcal{C} = \{\hat{\Sigma} \mid \|\Sigma^{-1/2}(\hat{\Sigma} - \Sigma)\Sigma^{-1/2}\| \leq \beta\}$$

- trace heuristic

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}\, X \\ \text{subject to} & X \succeq 0, \quad d \succeq 0 \\ & \|\Sigma^{-1/2}(X + \mathbf{diag}(d) - \Sigma)\Sigma^{-1/2}\| \leq \beta \end{array}$$

Trace approximation results