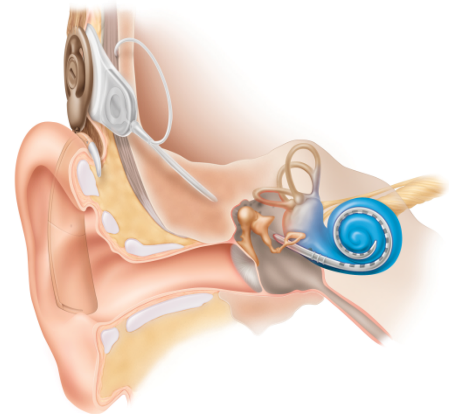
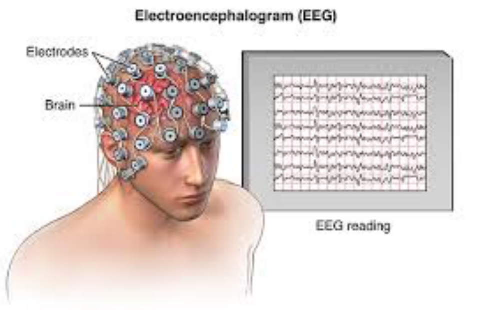


Stochastic processes
Exercise session

Dr. Ir. Richard C. Hendriks



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Estimation of a RV

RVs X and Y have the following joint pdf:

$$f_{X,Y}(x,y) = \begin{cases} 6(y-x) & 0 \leq x \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- What is $f_{X|Y}(x|y)$
- Give the MMSE estimate $\hat{x}_M(y)$ of X given $Y = y$.
- What is $f_{Y|X}(y|x)$
- Give the MMSE estimate $\hat{Y}_M(x)$ of Y given $X = x$.

Estimation of a RV

RVs X and Y have the following joint pdf:

$$f_{X,Y}(x,y) = \begin{cases} 6(y-x) & 0 \leq x \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- What is $f_{X|Y}(x|y)$?
- $f_Y(y) = \int_0^y 6(y-x)dx = \left[6xy - 3x^2\right]_0^y = 3y^2$, for $0 \leq y \leq 1$, otherwise $f_Y(y) = 0$.
- $$f_{X|Y}(x|y) = \begin{cases} \frac{6(y-x)}{3y^2} = \frac{2(y-x)}{y^2} & 0 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

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Estimation of a RV

- What is $E[X|Y]$?
-

$$E[X|Y] = \int_0^y x\frac{2(y-x)}{y^2}dx = \int_0^y \frac{2yx - 2x^2}{y^2}dx = \left[\frac{yx^2 - 2/3x^3}{y^2}\right]_0^y$$

$$\frac{y^3 - 2/3y^3}{y^2} = \frac{y}{3}$$

Estimation of a RV

RVs X and Y have the following joint pdf:

$$f_{X,Y}(x,y) = \begin{cases} 6(y-x) & 0 \leq x \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- What is $f_{Y|X}(y|x)$
- $f_X(x) = \int_x^1 6(y-x)dy = \left[3y^2 - 6xy\right]_x^1 = 3 - 6x + 3x^2$. So,

$$f_X(x) = \begin{cases} 3 - 6x + 3x^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- $$f_{Y|X}(y|x) = \begin{cases} \frac{2(y-x)}{1-2x+x^2} & x \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Estimation of a RV

- What is $E[Y|X]$?

- $$f_{Y|X}(y|x) = \begin{cases} \frac{2(y-x)}{1-2x+x^2} & x \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- $E[Y|X] = \int_x^1 y \frac{2(y-x)}{1-2x+x^2} dy = \left[\frac{(\frac{2}{3}y^3 - xy^2)}{1-2x+x^2} \right]_x^1 = \frac{\frac{2}{3} - x + \frac{1}{3}x^3}{1-2x+x^2}$
$= \frac{1}{3} \frac{2-3x-x^3}{1-2x+x^2} = \frac{1}{3} \frac{(x+2)(1-x)^2}{(1-x)^2} = \frac{x}{3} + \frac{2}{3}$

Conditional probability Models

RVs X and Y have pdf

$$f_{X,Y}(x,y) = \begin{cases} \frac{4x+2y}{3} & 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $A = \{Y \leq 1/2\}$

- $P[A]$?
- $f_{X,Y|A}(x,y)$
- $f_{X|A}(x)$?

Conditional probability Models

- $P[A] = \int_0^1 \int_0^{1/2} \frac{4x+2y}{3} dydx = 5/12$
- $$f_{X,Y|A}(x,y) \begin{cases} \frac{f_{X,Y|A}(x,y)}{P(A)} = \frac{\frac{4x+2y}{3}}{\frac{5}{12}} = \frac{16x+8y}{5} & 0 \leq x \leq 1,\ 0 \leq y \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$$

Conditional probability Models

- $$f_{X,Y|A}(x,y)\begin{cases} \frac{f_{X,Y|A}(x,y)}{P(A)} = \frac{\frac{4x+2y}{3}}{\frac{5}{12}} = \frac{16x+8y}{5} & 0 \leq x \leq 1,\ 0 \leq y \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$$

- $f_{X|A} = \int_0^{1/2} \frac{16x+8y}{5} dy = \left[\frac{16}{5}xy + \frac{4}{5}y^2\right]_0^{1/2} = \frac{8}{5}x + \frac{1}{5}$

Mid-term Question 2018

Consider the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \lambda e^{-\lambda x} & \text{for} \quad 0 \leq y \leq x \\ 0 & \text{otherwise.} \end{cases}$$

a) Calculate the PDFs $f_X(x)$ and $f_Y(y)$ and use these marginal PDFs to argue whether or not X and Y are dependent.

b) Calculate the MMSE estimator $\hat{Y} = E[Y|X]$.

c) Calculate the MMSE estimator $\hat{X} = E[X|Y]$.

Mid-term Question 2018

(2 p) a) If $x \geq 0$ then

$$f_X(x) = \int_0^x \lambda e^{-\lambda x} dy = x\lambda e^{-\lambda x}$$

otherwise $f_X(x) = 0$. Similarly, if $y \geq 0$ then

$$f_Y(y) = \int_y^\infty \lambda e^{-\lambda x} dx = \left[-e^{-\lambda x}\right]_y^\infty = e^{-\lambda y}$$

otherwise $f_Y(y) = 0$.

RVs X and Y are dependent, as $f_{X,Y} \neq f_X f_Y$.

Mid-term Question 2018

(2 p) b)

$$f_{Y|X}(y|x) = \begin{cases} f_{X,Y}/f_X = \frac{1}{x} & \text{for} \quad 0 \leq y \leq x \\ 0 & \text{otherwise.} \end{cases}$$

$$\hat{Y} = E[Y|X] = \int_0^x y\frac{1}{x}dy = \frac{x}{2}\,.$$

Mid-term Question 2018

(2 p) c) Calculate the MMSE estimator $\hat{X} = E[X|Y]$

$$f_{X|Y}(x|y) = \begin{cases} f_{X,Y}/f_Y = \lambda e^{-\lambda(x-y)} & \text{for} \quad y \leq x \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \hat{X} &= E[X|Y] = \int_y^\infty x\lambda e^{-\lambda(x-y)} dx \\ &= \lambda e^{\lambda y} \left(\left[-\frac{1}{\lambda} x e^{-\lambda x} \right]_y^\infty + \int_y^\infty \frac{1}{\lambda} e^{-\lambda x} dx \right) \\ &= y + \frac{1}{\lambda} \end{aligned}$$

Mid-term Question 2018

(2 p) c) Or, using the MGF:

$$\phi_{X|Y}(s) = \frac{\lambda e^{ys}}{\lambda - s}$$

$$\left. \frac{d\frac{\lambda e^{ys}}{\lambda - s}}{ds} \right|_{s=0} = y + \frac{1}{\lambda}$$

Bounds on Probabilities

Let $R_n = \sum_{m=1}^{n} K_m$ with $K_m \geq 0$ a RV with $E[K_m] = 2$.

1. Give the value for n such that $P(R_n > 60) \leq \frac{1}{10}$

Bounds on Probabilities

Let $R_n = \sum_{m=1}^{n} K_m$ with $K_m \geq 0$ a RV with $E[K_m] = 2$.

1. Give the value for n such that $P(R_n > 60) \leq \frac{1}{10}$
   Using Markov inequality: $P(R_n > 60) \leq \frac{E[R_n]}{60} = \frac{n2}{60} \leq \frac{1}{10}$
   From this it follows that $n \leq 3$

How can we say more about $P(R_n > 60)$? Calculate the PMF.

From PMF to MGF and back...

Given is a RV K_n with PMF

$$P_{K_n} = \begin{cases} 2^k e^{-2}/k! & k = 0, 1, 2, ..., \\ 0 & \text{otherwise} \end{cases}$$

1. Calculate the moment generating function $\Phi_{K_n}(s)$.
2. Let $R_n = \sum_{m=1}^{n} K_m$. What is the MGF ϕ_{R_n}?
3. Determine the PMF P_{R_n}.

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From PMF to MGF and back...

1. a) $\Phi_{K_n}(s) = \sum_{k=0}^{\infty} \frac{2^k e^{-2}}{k!} e^{sk} = e^{-2} \sum_{k=0}^{\infty} \frac{(2e^s)^k}{k!}$
   Remember Taylor series for e^x: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
   Thus, $\Phi_{K_n}(s) = e^{-2} e^{2e^s} = e^{2(e^s-1)}$

   b) Using Table 9.1 p. 295: $P_{K_i}(k)$ is a Poisson distribution with $\alpha = 2$, with MGF $\Phi_{K_n}(s) = e^{\alpha(e^s-1)} = e^{2(e^s-1)}$.

2. $\phi_{R_n} = \prod_{m=1}^{n} e^{2(e^s-1)} = e^{2n(e^s-1)}$

3. We know that $\phi_{R_i} = e^{2n(e^s-1)}$. This is the MGF of again a Poisson distributed RV with $\alpha = 2n$:

$$P_{R_n}(r) = \begin{cases} (2n)^r e^{-2n}/r! & r = 0, 1, 2, ..., \\ 0 & \text{otherwise} \end{cases}$$

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Chernoff Bound

Let $R_n = \sum_{m=1}^{n} K_m$ with $K_m \geq 0$ a RV with $E[K_m] = 2$.

1. Give the value for n such that $P(R_n > 60) \leq \frac{1}{10}$

Chernoff Bound

Let $R_n = \sum_{m=1}^{n} K_m$ with $K_m \geq 0$ a RV with $E[K_m] = 2$.

1. Give the value for n such that $P(R_n > 60) \leq \frac{1}{10}$

$$P(R_n > 60) \leq \min_{s \geq 0} e^{-s60} \phi_{R_n}(s) = \min_{s \geq 0} e^{-s60} e^{2n(e^s - 1)}$$

$$\frac{d(-s60 + 2n(e^s - 1))}{ds} = -60 + 2ne^s = 0 \;\Rightarrow\; s = \log\left(\frac{60}{2n}\right)$$

Note: s should be $s \geq 0 \Rightarrow$ max. value n for which bound is valid is $n = 30$.

$$P(R_n > 60) \leq e^{-s60} e^{2n(e^s - 1)}\Big|_{s=\log\left(\frac{60}{2n}\right)} = e^{-60\log\left(\frac{60}{2n}\right)} e^{2n\left(\left(\frac{60}{2n}\right) - 1\right)} =$$

$$e^{-60\log\left(\frac{60}{2n}\right) + 60 - 2n}$$

For $n = 3$ we get according to Chernoff $P(R_n \geq 60) \leq 2.8 \cdot 10^{-37}$

Using the derived PMF: $P(R_n \geq 60) = 1.5 \cdot 10^{-38}$