EE2S31 Signal Processing – Stochastic Processes

Lecture 9: Exercises for Part 2

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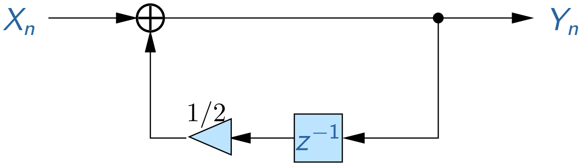
Exam 27 July 2021, Question 3

Consider the following discrete-time system:



The input signal is an iid Gaussian random process X_n, with mean $\mu_X = 2$ and variance $\sigma_X^2 = 3$.

The output Y_n satisfies the recursion $Y_n = \frac{1}{2} Y_{n-1} + X_n$.

(a) Determine the autocorrelation sequence of the input, $R_X[k]$, as well as its power spectral density, $S_X(\phi)$.

(b) Compute $\mathrm{E}[Y_n]$.

Exam 27 July 2021

Solution

(a) The input is iid (hence WSS), and

$$R_X[k] = \sigma_X^2 \delta[k] + \mu_X^2 = 3\delta[k] + 4\,.$$

The input power spectral density is the DTFT of $R_X[k]$, i.e.,

$$S_X(\phi) = \sigma_X^2 + \mu_X^2 \delta(\phi) = 3 + 4\delta(\phi)\,.$$

(b) Using the recursion gives $\mathrm{E}[Y_n] = \frac{1}{2}\mathrm{E}[Y_{n-1}] + \mathrm{E}[X_n]$. Since Y_n is WSS (output of an LTI filter with WSS process as input), $\mathrm{E}[Y_n] = \mathrm{E}[Y_{n-1}] = \mu_Y$, and we find

$$\mu_Y = \frac{1}{2}\mu_Y + \mu_X \qquad \Rightarrow \qquad \mu_Y = 2\mu_X = 4\,.$$

Alternatively, use $\mu_Y = \mu_X \sum_n h[n]$, with $h[n] = (\frac{1}{2})^n u[n]$. Then $\sum_n h[n] = \frac{1}{1-1/2} = 2$, and $\mu_Y = 4$.

Exam 27 July 2021, Question 3
(Continued)

The autocovariance sequence of the output is

$$C_Y[k] = \frac{4}{3}\left(\frac{1}{2}\right)^{|k|} \sigma_X^2 \, .$$

(c) Compute the autocorrelation sequence $R_Y[k]$ of the output.

(d) What is the average output power?

(e) Determine the power spectral density of the output, $S_Y(\phi)$.

Note: See Table 3 (Suppl. page 38) for Discrete-Time Fourier Transform pairs.

Exam 27 July 2021, Question 3

Solution

(c) $R_Y[k] = C_Y[k] + \mu_Y^2 = 4\left(\frac{1}{2}\right)^{|k|} + 16.$

(d) $R_Y[0] = 20.$

(e) Take the DTFT of $R_Y[k]$. Using Table 3 (Suppl. page 38),

$$S_Y(\phi) = \frac{4}{3}\frac{1-\frac{1}{4}}{1+\frac{1}{4}-\cos(2\pi\phi)}\,\sigma_X^2 + \mu_Y^2\delta(\phi) = \frac{3}{\frac{5}{4}-\cos(2\pi\phi)} + 16\delta(\phi)$$

Alternatively, use $H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$ and evaluate $S_Y(\phi) = |H(e^{j2\pi\phi})|^2 S_X(\phi)$:

$$\begin{aligned} S_Y(\phi) &= \frac{1}{1-\frac{1}{2}e^{-j2\pi\phi}}\frac{1}{1-\frac{1}{2}e^{j2\pi\phi}}(3+4\delta(\phi)) \\ &= \frac{1}{1+\frac{1}{4}-\frac{1}{2}e^{-j2\pi\phi}-\frac{1}{2}e^{j2\pi\phi}}(3+4\delta(\phi)) \\ &= \frac{3}{\frac{5}{4}-\cos(2\pi\phi)} + 16\delta(\phi) \end{aligned}$$

Exam July 1, 2021, Question 1

Let $X(t) = A\cos(\Omega_0 t)$ be a random process, where $A \in \{-1, +1\}$ with equal probabilities, and Ω_0 is a given frequency.

(a) Draw two different realizations of $X(t)$.

(b) What type of random process is $X(t)$? [Think of continuous value/discrete value; continuous-time/discrete time.]

(c) Compute the probability mass function (PMF) $P_{X(t)}(x)$.

Exam July 1, 2021, Question 1

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(b) What type of random process is $X(t)$? [Think of continuous value/discrete value; continuous-time/discrete time.]
(c) Compute the probability mass function (PMF) $P_{X(t)}(x)$.

(a) (There are only 2 possibilities, one for $A = 1$, the other for $A = -1$)
(b) This is a discrete value continuous-time random process. (Therefore, $X(t)$ is described by a PMF.)
(c)

$$P_{X(t)}(x) = \begin{cases} \frac{1}{2} & x = \cos(\Omega_0 t) \\ \frac{1}{2} & x = -\cos(\Omega_0 t) \\ 0 & \text{otherwise} \end{cases}$$

Exam July 1, 2021, Question 1

(Continued)

(d) Compute $E[X(t)]$.

(e) Compute $R_X(t, \tau)$.

(f) Is $X(t)$ stationary? Is it WSS?

Exam July 1, 2021, Question 1

(Continued)

(d) Compute $E[X(t)]$.

(e) Compute $R_X(t,\tau)$.

(f) Is $X(t)$ stationary? Is it WSS?

(d) $E[X(t)] = E[A]\cos(\Omega_0 t) = 0$.

(e) Note that $E[A^2] = 1$. Then

$$\begin{aligned} R_X(t,\tau) &= E[A\cos(\Omega_0 t) A\cos(\Omega_0(t+\tau))] \\ &= E[A^2]\cos(\Omega_0 t)\cos(\Omega_0(t+\tau)) \\ &= \frac{1}{2}\cos(\Omega_0\tau) + \frac{1}{2}\cos(2\Omega_0 t + \Omega_0\tau) \end{aligned}$$

(f) Not stationary because $P_{X(t)}(x) \neq P_{X(t+\tau)}(x)$.
Not WSS because $R_X(t,\tau)$ depends on t.

Exam July 1, 2021, Question 2

Let X_n be an independent identically distributed (iid) random sequence with mean 2 and variance 3, and consider $Y_n = \frac{1}{2}(X_n + X_{n-1})$.

(a) Compute $\mathrm{E}[Y_n]$.

(b) Compute $\mathrm{var}[Y_n]$.

Exam July 1, 2021, Question 2

Let X_n be an independent identically distributed (iid) random sequence with mean 2 and variance 3, and consider $Y_n = \frac{1}{2}(X_n + X_{n-1})$.

(a) Compute $\mathrm{E}[Y_n]$.

(b) Compute $\mathrm{var}[Y_n]$.

(a) Use independence of X_n and X_{n-1}:
$\mathrm{E}[Y_n] = \frac{1}{2}(\mathrm{E}[X_n] + \mathrm{E}[X_{n-1}) = 2$.

(b) Use independence of X_n and X_{n-1}:
$\mathrm{var}[Y_n] = \frac{1}{4}(\mathrm{var}(X_n) + \mathrm{var}(X_{n-1})) = \frac{3}{2}$.

Exam July 1, 2021, Question 2
(Continued)

(c) Compute $R_X[k]$.

Exam July 1, 2021, Question 2
(Continued)

(c) Compute $R_X[k]$.

(c) The extended derivation is, using iid,

$$\begin{aligned} R_X[k] &= \mathrm{E}[X_n X_{n+k}] = \begin{cases} \mathrm{E}[X_n^2] & k = 0 \\ \mathrm{E}[X_n]\mathrm{E}[X_{n+k}] & k \neq 0 \end{cases} \\ &= \begin{cases} \mu_X^2 + \mathrm{var}[X_n] & k = 0 \\ \mu_X^2 & k \neq 0 \end{cases} \\ &= \begin{cases} 4 + 3 & k = 0 \\ 4 & k \neq 0 \end{cases} \end{aligned}$$

Write this in one expression as $R_X[k] = 4 + 3\delta[k]$.

Exam July 1, 2021, Question 2

(Continued)

(d) Compute $R_{XY}[n, k]$ and $R_Y[n, k]$.

Exam July 1, 2021, Question 2
(Continued)

(d) Compute $R_{XY}[n,k]$ and $R_Y[n,k]$.

(d)
$$
\begin{aligned}
R_{XY}[n,k] &= \mathrm{E}[X_n Y_{n+k}] = \frac{1}{2}\mathrm{E}[X_n(X_{n+k} + X_{n+k-1})] \\
&= \frac{1}{2}(R_X[k] + R_X[k-1]) \;=\; 4 + \frac{3}{2}\delta[k] + \frac{3}{2}\delta[k-1] \\
R_Y[n,k] &= \mathrm{E}[Y_n Y_{n+k}] = \frac{1}{4}\mathrm{E}[(X_n + X_{n-1})(X_{n+k} + X_{n+k-1})] \\
&= \frac{1}{4}(\mathrm{E}[X_n X_{n+k}] + \mathrm{E}[X_n X_{n+k-1}] \\
&\qquad + \mathrm{E}[X_{n-1}X_{n+k}] + \mathrm{E}[X_{n-1}X_{n+k-1}]) \\
&= \frac{1}{4}(2R_X[k] + R_X[k-1] + R_X[k+1]) \\
&= 4 + \frac{3}{2}\delta[k] + \frac{3}{4}\delta[k-1] + \frac{3}{4}\delta[k+1]
\end{aligned}
$$

Alternatively, use the convolution equations.

Exam July 1, 2021, Question 2

(Continued)

(e) Compute the average power of Y_n.

(f) Is Y_n iid? Is it WSS? Is it jointly WSS with X_n?

(g) If X_n is Gaussian, is Y_n Gaussian?

Exam July 1, 2021, Question 2
(Continued)

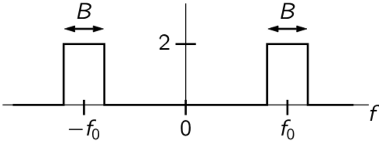
(e) Compute the average power of Y_n.
(f) Is Y_n iid? Is it WSS? Is it jointly WSS with X_n?
(g) If X_n is Gaussian, is Y_n Gaussian?

(e) $\mathrm{E}[Y_n^2] = R_Y[0] = 4 + \frac{3}{2} = 5.5$
(f) Not iid because Y_n is not independent of Y_{n-1} (they both depend on X_{n-1}). This is also seen from $R_Y[k]$ or, more clearly, from $C_Y[k] = R_Y[k] - \mu_Y^2 = \frac{3}{2}\delta[k] + \frac{3}{4}\delta[k-1] + \frac{3}{4}\delta[k+1]$: for an iid process we would only have a term with $\delta[k]$.
WSS because $\mathrm{E}[Y_n]$ does not depend on n and $R_Y[n,k]$ does not depend on n.
Jointly WSS because both X_n and Y_n are WSS, and $R_{XY}[n,k]$ does not depend on n.
(g) Yes, Y_n is also Gaussian distributed, because it is a linear combination of Gaussian variables.

Exam July 1, 2021, Question 3

The power spectral density $S_X(f)$ of a random process $X(t)$ is given by

$$S_X(f) = \begin{cases} 2 & |f \pm f_0| \leq \frac{B}{2} \\ 0 & \text{otherwise} \end{cases}$$



(a) Compute the average power of $X(t)$.

(b) Determine the autocorrelation function $R_X(\tau)$.
Hint: You may need to use Supplement table 1, 2, p. 29/30.

Exam July 1, 2021, Question 3

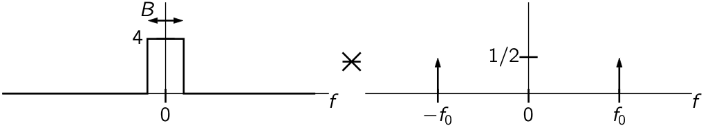
Solution

(a) The average power is the area in the figure:

$$\mathrm{E}[X^2(t)] = R_X(0) = \int_{-\infty}^{\infty} S_X(f)\mathrm{d}f = 4B$$

(b) First recognize that $S_X(f)$ is the convolution of a baseband lowpass filter with two delta pulses in frequency:

$$S_X(f) = S_B(f) * C(f)$$



Exam July 1, 2021, Question 3

Solution (continued)

(b) The autocorrelation function $R_X(\tau)$ is the inverse Fourier transform of $S_X(f)$, hence (see table)

$$R_X(\tau) = R_B(\tau)\, c(\tau)$$

Next use the table:

$$\begin{aligned} \text{sinc}(2W\tau) \quad &\leftrightarrow \quad \frac{1}{2W}\text{rect}\left(\frac{f}{2W}\right) \\ \cos(2\pi f_0 \tau) \quad &\leftrightarrow \quad \frac{1}{2}\left(\delta(f - f_0) + \delta(f + f_0)\right) \end{aligned}$$

Note that $B = 2W$. Altogether, this gives

$$R_X(\tau) = 4B\,\text{sinc}(B\tau)\,\cos(2\pi f_0 \tau)$$

(You can check the scale by evaluating $R_X(0) = 4B$, and compare to question (a).)

Exam July 1, 2021, Question 3
(Continued)

(c) $X(t)$ can be generated by passing white noise through a filter. Assume the noise power spectral density of the input is 1 W/Hz. Specify the filter transfer function $H(f)$.

(d) Let $Y(t) = X(t-5)$. Determine $S_Y(f)$.

Exam July 1, 2021, Question 3

(Continued)

(c) $X(t)$ can be generated by passing white noise through a filter. Assume the noise power spectral density of the input is 1 W/Hz. Specify the filter transfer function $H(f)$.

(d) Let $Y(t) = X(t-5)$. Determine $S_Y(f)$.

(c) The filter $H(f)$ needs to satisfy $|H(f)|^2 = S_X(f)$. Hence, it is a bandpass filter,

$$H(f) = \begin{cases} \sqrt{2} & |f \pm f_0| \leq \frac{B}{2} \\ 0 & \text{otherwise} \end{cases}$$

where in fact the phase is arbitrary.

(d) The delay in time domain corresponds to a phase shift in frequency domain. This is a filter $G(f)$ with $|G(f)|^2 = 1$. Since $S_Y(f) = |G(f)|^2 S_X(f)$, we have $S_Y(f) = S_X(f)$: the same.

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(Continued)

Let $Z(t) = 2\,X(t) + N(t)$, where $N(t)$ is independent white noise with power spectral density N_0.

(e) Determine $S_Z(f)$.
(f) Determine $S_{XZ}(f)$ and $S_{NZ}(f)$.

Exam July 1, 2021, Question 3

(Continued)

Let $Z(t) = 2\,X(t) + N(t)$, where $N(t)$ is independent white noise with power spectral density N_0.

(e) Determine $S_Z(f)$.
(f) Determine $S_{XZ}(f)$ and $S_{NZ}(f)$.

(e) The power spectral density of the noise is $S_N(f) = N_0$ (a constant). Then, since the noise is independent,

$$S_Z(f) = 4S_X(f) + S_N(f) = \begin{cases} 8 + N_0 & |f \pm f_0| \leq \frac{B}{2} \\ N_0 & \text{otherwise} \end{cases}$$

(f)

$$\begin{aligned} R_{XZ}(\tau) &= \mathrm{E}[X(t)Z(t+\tau)] = \mathrm{E}[X(t)(2X(t+\tau) + N(t+\tau)] \\ &= 2\mathrm{E}[X(t)(X(t+\tau)] = 2R_X(\tau) \end{aligned}$$

Therefore: $S_{XZ}(f) = 2S_X(f)$. Similarly, $S_{NZ}(f) = S_N(f) = N_0$.

Exam July 31, 2020, Question 3

Let W be an exponentially distributed random variable, with pdf

$$f_W(w) = \begin{cases} e^{-w} & w \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Consider the random process $X(t) = t - W$.

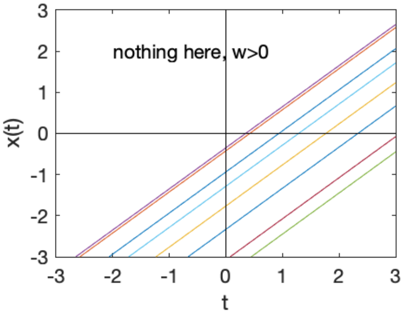
(a) Draw three realizations of $X(t)$.

(b) Determine the CDF $F_W(w)$ and $F_{X(t)}(x)$, and the pdf of $X(t)$.

Exam July 31, 2020, Question 3

Solution

(a) Pick w following an exponential distribution (hence $w > 0$, then plot $X(t) = t - w$ as function of t:



Exam July 31, 2020, Question 3
(Continued)

(b) W has an exponential distribution with $\lambda = 1$. (See Thm. 4.8)

$$F_W(w) = \mathrm{P}[W < w] = \begin{cases} 1 - e^{-w} & w \geq 0 \\ 0 & w < 0 \end{cases}$$

$$\begin{aligned} F_{X(t)}(x) &= \mathrm{P}[X < x] = \mathrm{P}[t - W < x] = \mathrm{P}[W > t - x] \\ &= 1 - \mathrm{P}[W < t - x] = \begin{cases} e^{x-t} & x \leq t \\ 1 & x > t \end{cases} \end{aligned}$$

$$f_{X(t)}(x) = \frac{\mathrm{d}\, F_{X(t)}(x)}{\mathrm{d}\, x} = \begin{cases} e^{x-t} & x \leq t\,, \\ 0 & x > t \end{cases}$$

Exam July 31, 2020, Question 3

(Continued)

(c) Determine $E[W]$ and compute the expected value function, $\mu_{X(t)}$.

(d) Determine $E[W^2]$ and compute the autocovariance function, $C_X(t, \tau)$.

Exam July 31, 2020, Question 3

(Continued)

(c) Determine $\mathrm{E}[W]$ and compute the expected value function, $\mu_{X(t)}$.

(d) Determine $\mathrm{E}[W^2]$ and compute the autocovariance function, $C_X(t,\tau)$.

(c) For an exponential distribution with $\lambda = 1$, $\mathrm{E}[W] = 1$

$$\mu_X(t) = \mathrm{E}[t - W] = t - \mathrm{E}[W] = t - 1\,.$$

(d) $\mathrm{E}[W^2] = 2$.

$$\begin{aligned}
C_X(t,\tau) &= \mathrm{E}[X(t)X(t+\tau)] - \mu_X(t)\mu_X(t+\tau) \\
&= \mathrm{E}[(t-W)(t+\tau-W)] - (t-1)(t+\tau-1) \\
&= t(t+\tau) - \mathrm{E}[(2t+\tau)W] + \mathrm{E}[W^2] \\
&\qquad -t(t+\tau) + 2t + \tau - 1 \\
&= -(2t+\tau)\mathrm{E}[W] + 2 + 2t + \tau - 1 \\
&= 1
\end{aligned}$$

Exam July 31, 2020, Question 3
(Continued)

(e) Is $X(t)$ a WSS random process? (Motivate)

(f) Is it an i.i.d. process? (Motivate)

Exam July 31, 2020, Question 3

(Continued)

(e) Is $X(t)$ a WSS random process? (Motivate)

(f) Is it an i.i.d. process? (Motivate)

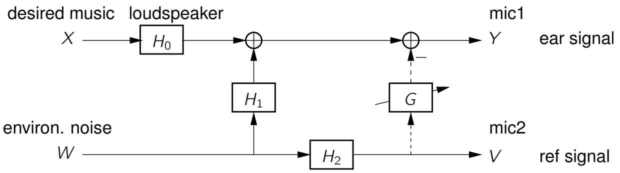
(e) Not WSS because $\mu_X(t)$ is dependent on t.

(f) Not iid because samples are clearly correlated to each other (not independent) and samples do not have the same distribution (it depends on t).

Exam July 31, 2020, Question 4

In this question, all signals are considered in the frequency domain.

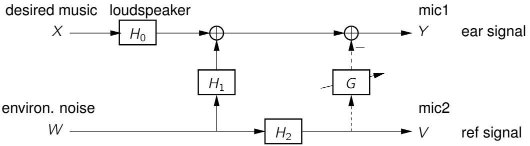
The fundamentals behind a noise canceling headphone are schematically drawn in the figure.



We wish to listen to music $X(f)$ transmitted over a loudspeaker with unknown response $H_0(f)$, but the ear signal $Y(f)$ is disturbed by unknown environment noise $W(f)$, which has been filtered by an unknown channel response $H_1(f)$. We measure the ear signal with microphone 1. An additional microphone (mic2) also captures the noise signal, but it is filtered by an unknown filter $H_2(f)$.

Exam July 31, 2020, Question 4

(Continued)

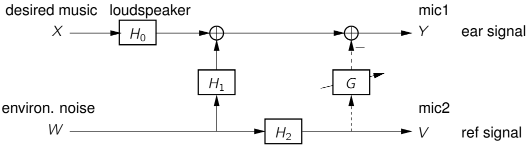


We wish to design a filter $G(f)$ such that the noise signal on Y is perfectly canceled. **While we design $G(f)$, it is not included in the schematic.**

(a) What is the desired solution for $G(f)$ in terms of $H_1(f)$ and $H_2(f)$?
(b) Show that $H_2^{-1}(f) = |H_2(f)|^{-2} H_2^*(f)$.

Exam July 31, 2020, Question 4

(Continued)



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(a) What is the desired solution for $G(f)$ in terms of $H_1(f)$ and $H_2(f)$?
(b) Show that $H_2^{-1}(f) = |H_2(f)|^{-2} H_2^*(f)$.

(a) $G(f) = H_2^{-1}(f) H_1(f)$.
(b) It is the inverse because
$H_2^{-1}(f) H_2(f) = |H_2(f)|^{-2} H_2^*(f) H_2(f) = 1$.

Exam July 31, 2020, Question 4

(Continued)

$X(f)$ and $W(f)$ are considered to be independent random processes, with power spectral densities $S_X(f)$ and $S_W(f)$, respectively.

(c) Give expressions for $S_Y(f)$, $S_V(f)$ and $S_{YV}(f)$ in terms of $S_X(f)$ and $S_W(f)$.

(d) Which of these (cross) power spectral densities can we observe?

(e) Give an expression for $G(f)$ in terms of observed quantities.

Exam July 31, 2020, Question 4

(Continued)

$X(f)$ and $W(f)$ are considered to be independent random processes, with power spectral densities $S_X(f)$ and $S_W(f)$, respectively.

(c) Give expressions for $S_Y(f)$, $S_V(f)$ and $S_{YV}(f)$ in terms of $S_X(f)$ and $S_W(f)$.

(d) Which of these (cross) power spectral densities can we observe?

(e) Give an expression for $G(f)$ in terms of observed quantities.

(c)

$$\begin{aligned} S_Y(f) &= |H_0(f)|^2 S_X(f) + |H_1(f)|^2 S_W(f) \\ S_V(f) &= |H_2(f)|^2 S_W(f) \\ S_{YV}(f) &= H_1(f) H_2^*(f) S_W(f) \end{aligned}$$

(d) Using the microphone signals, we can observe $Y(f)$ and $V(f)$, and estimate $S_Y(f)$, $S_V(f)$ and $S_{YV}(f)$.

(e)

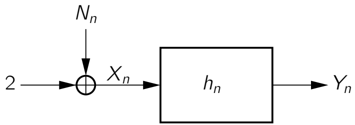
$$G(f) = S_V^{-1}(f) S_{YV}(f)$$

Exam 2 July 2020, Question 4

Let the random sequence X_n be a constant 2, perturbed by zero mean i.i.d. noise N_n, with $\mathrm{var}[N_n] = \sigma^2$.

The random sequence Y_n is obtained by filtering X_n, where the impulse response h_n of the LTI filter is given by

$$h_n = \begin{cases} 1 & n = 0, \\ -\frac{1}{2} & n = 1, \\ 0 & \text{otherwise.} \end{cases}$$



Exam 2 July 2020, Question 4

(a) Show that the auto-correlation sequence of X_n is given by

$$R_X[k] = 4 + \sigma^2 \delta[k] \,.$$

(b) Find $\mathrm{E}[Y_n]$.

Exam 2 July 2020, Question 4

(a) Show that the auto-correlation sequence of X_n is given by

$$R_X[k] = 4 + \sigma^2 \delta[k] \,.$$

(b) Find $\mathrm{E}[Y_n]$.

(a) Since X_n is i.i.d., we know $C_X[k] = \sigma^2 \delta[k]$. Then

$$R_X[k] = C_X[k] + \mathrm{E}[X]^2 = \sigma^2 \delta[k] + 4 \,.$$

(b)

$$Y_n = X_n - \frac{1}{2} X_{n-1}$$

$$\mathrm{E}[Y_n] = \mathrm{E}[X_n] - \frac{1}{2} \mathrm{E}[X_{n-1}] = 2 - 1 = 1 \,.$$

Exam 2 July 2020, Question 4

(Continued)

(c) Find the auto-correlation $R_Y[n, k]$ and the auto-covariance $C_Y[n, k]$.

Exam 2 July 2020, Question 4

(Continued)

(c) Find the auto-correlation $R_Y[n,k]$ and the auto-covariance $C_Y[n,k]$.

(c)
$$
\begin{aligned}
R_Y[n,k] &= \mathrm{E}[Y_n Y_{n+k}] = \mathrm{E}[(X_n - \frac{1}{2}X_{n-1})(X_{n+k} - \frac{1}{2}X_{n+k-1})] \\
&= \mathrm{E}[X_n X_{n+k}] - \frac{1}{2}\mathrm{E}[X_{n-1}X_{n+k}] - \frac{1}{2}\mathrm{E}[X_n X_{n+k-1}] + \frac{1}{4}\mathrm{E}[X_{n-} \\
&= 4 + \sigma^2\delta[k] - \frac{1}{2}\left(4 + \sigma^2\delta[k+1]\right) - \frac{1}{2}\left(4 + \sigma^2\delta[k-1]\right) + \\
&= 1 + \sigma^2\left(\frac{5}{4}\delta[k] - \frac{1}{2}\delta[k+1] - \frac{1}{2}\delta[k-1]\right) \\
C_Y[n,k] &= R_Y[n,k] - \mathrm{E}[Y_n]\mathrm{E}[Y_{n+k}] \\
&= \sigma^2\left(\frac{5}{4}\delta[k] - \frac{1}{2}\delta[k+1] - \frac{1}{2}\delta[k-1]\right)
\end{aligned}
$$

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(Continued)

(d) Is Y_n i.i.d.? Is Y_n wide sense stationary? (Motivate)

(e) Find the cross-correlation $R_{XY}[n, k]$ and cross-covariance $C_{XY}[n, k]$.

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(Continued)

(d) Is Y_n i.i.d.? Is Y_n wide sense stationary? (Motivate)

(e) Find the cross-correlation $R_{XY}[n, k]$ and cross-covariance $C_{XY}[n, k]$.

(d) Not i.i.d.: $C_Y[n, k]$ shows clearly that Y_n is not independent from Y_{n-1}. WSS because $\mathrm{E}[Y_n]$ is independent of n and $C_Y[n, k]$ is independent of n.

(e)
$$\begin{aligned}
R_{XY}[n,k] &= \mathrm{E}[X_n Y_{n+k}] = \mathrm{E}[X_n(X_{n+k} - \frac{1}{2}X_{n+k-1})] \\
&= R_X[k] - \frac{1}{2}R_X[k-1] = 4 + \sigma^2\delta[k] - \frac{1}{2}\left(4 + \sigma^2\delta[k-1]\right) \\
&= 2 + \sigma^2\left(\delta[k] - \frac{1}{2}\delta[k-1]\right)
\end{aligned}$$

$$C_{XY}[n,k] = \sigma^2\left(\delta[k] - \frac{1}{2}\delta[k-1]\right)$$

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(Continued)

(f) Are X_n and Y_n jointly wide sense stationary? (Motivate)

(g) Compute the average power of Y_n.

(h) If, moreover, N_n is Gaussian distributed, then is Y_n Gaussian distributed? (Motivate)

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(Continued)

(f) Are X_n and Y_n jointly wide sense stationary? (Motivate)

(g) Compute the average power of Y_n.

(h) If, moreover, N_n is Gaussian distributed, then is Y_n Gaussian distributed? (Motivate)

(f) Yes, because X_n and Y_n are each WSS, and $C_{XY}[n, k]$ only depends on k.

(g) The average power is $\mathrm{E}[Y_n^2] = R_Y[0] = 1 + \frac{5}{4}\sigma^2$.

(h) Yes because the sum of Gaussian random variables is again a Gaussian random variable.