EE2S31 Signal Processing – Stochastic Processes

Lecture 8: Frequency Domain Relationships – Suppl. 7, 8

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Summarizing: Power spectral density

For WSS random processes we use the power spectral density to provide a frequency domain description.

Time-continuous:

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} \mathrm{d}\tau$$

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f\tau} \mathrm{d}f$$

Time-discrete:

$$S_X(\phi) = \sum_{k-\infty}^{\infty} R_X[k] e^{-j2\pi\phi k}$$

$$R_X[k] = \int_{-1/2}^{1/2} S_X(\phi) e^{j2\pi\phi k} \mathrm{d}\phi$$

- $S_X(\phi) \geq 0$ for all f
- $\int_{-1/2}^{1/2} S_X(\phi)\mathrm{d}\phi = \mathrm{E}[X_n^2] = R_X[0]$
- $S_X(-\phi) = S_X(\phi)$
- for any integer n, $S_X(\phi + n) = S_X(\phi)$ (periodic.)

Cross power spectral density

- The **cross-correlation** between two stochastic processes is defined as

$$R_{XY}(t,\tau) = \mathrm{E}[X(t)Y(t+\tau)]$$

Two random processes $X(t)$ and $Y(t)$ are jointly wide sense stationary, if $X(t)$ and $Y(t)$ are wide sense stationary, and

$$R_{XY}(t,\tau) = R_{XY}(\tau).$$

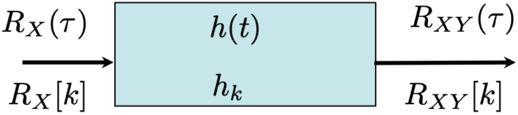
If $X(t)$ and $Y(t)$ are jointly WSS, then $R_{XY}(\tau) = R_{YX}(-\tau)$.

- Define the **cross power spectral density** for jointly WSS processes $X(t)$ and $Y(t)$:

$$S_{XY}(f) = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j2\pi f\tau}\mathrm{d}\tau$$

(Similar for time-discrete processes)

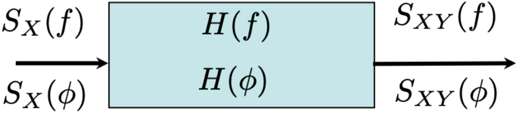
Frequency Domain Relationships I



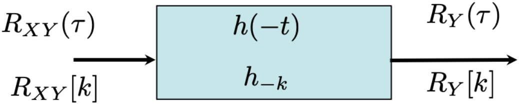
We know: $R_{XY}(\tau) = h(\tau) * R_X(\tau)$ and $R_{XY}[k] = h[k] * R_X[k]$

$\Leftarrow$Fourier transform$\Rightarrow$

$$S_{XY}(f) = H(f)S_X(f) \text{ and } S_{XY}(\phi) = H(\phi)S_X(\phi)$$



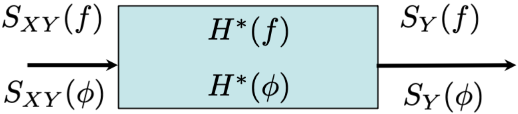
Frequency Domain Relationships II



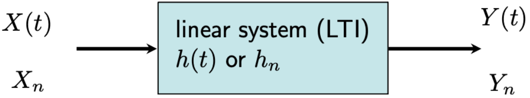
We know: $R_Y(\tau) = h(-\tau) * R_{XY}(\tau)$ and $R_Y[k] = h[-k] * R_{XY}[k]$

$\Leftarrow$Fourier transform$\Rightarrow$

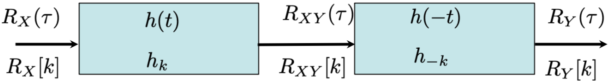
$$S_Y(f) = H^*(f)S_{XY}(f) \text{ and } S_Y(\phi) = H^*(\phi)S_{XY}(\phi)$$



Summary



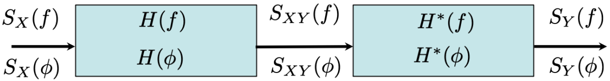
Time domain:



$$S_Y(f) = H^*(f)S_{XY}(f) = |H(f)|^2 S_X(f)$$

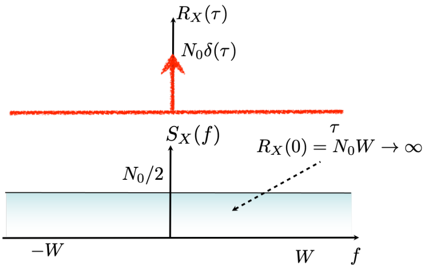
$$S_Y(\phi) = H^*(\phi)S_{XY}(\phi) = |H(\phi)|^2 S_X(\phi)$$

Frequency domain:



Continuous Time White Noise Process

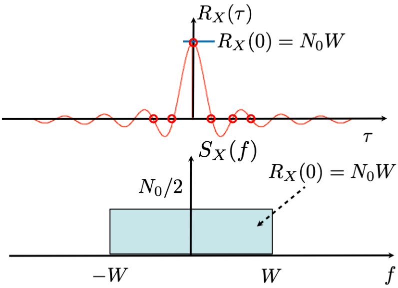
- Let $X(t)$ be a white noise process (i.e., zero mean and uncorrelated), with $R_x(\tau) = N_0\delta(\tau)$
- Then $S_X(f) = N_0$: constant for all f
- What is the average power of $X(t)$?



The average power is $R_X(0) = N_0W \to \infty$ as the bandwidth $W \to \infty$
This process cannot be physically realized (infinite average power)

Continuous Time Bandlimited White Noise Process

- What happens if we bandlimit the process?

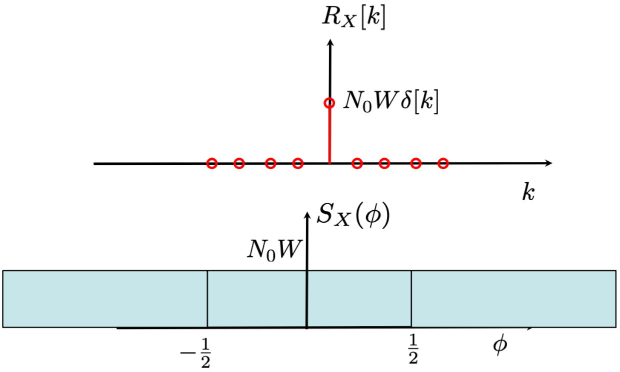


From Table 1:
$N_0 W \text{sinc}(2W\tau) \quad \Leftrightarrow \quad \frac{N_0}{2} \text{rect}\left(\frac{f}{2W}\right)$

zero for $\tau = \frac{k}{2W}$ $(k \in \mathbb{Z} \backslash \{0\})$

White Noise Process for Discrete-time signals

Sampling at $f_s = 2W$, the resulting discrete-time random process is truly white, with $R_X[k] = N_0 W \delta[k]$



(Recall sampling: $X_s(\Omega) = \frac{1}{T_s} \sum X(\Omega - k\Omega_s)$)

Problem 8.2 (really the same... with $B = 2W$)

Let $W(t)$ denote a WSS Gaussian noise process with $\mu_W = 0$ and power spectral density $S_W(f) = 1$.

(a) What is $R_W(\tau)$, the autocorrelation of $W(t)$?

(b) $W(t)$ is the input to a linear time-invariant filter with impulse response

$$H(f) = \begin{cases} 1 & |f| \leq B/2 \\ 0 & \text{otherwise} \end{cases}$$

The filter output is $Y(t)$. What is the power spectral density function of $Y(t)$?

(c) What is the average power of $Y(t)$?

(d) What is the expected value of the filter output?

Problem 8.2 (really the same... with $B = 2W$)

(a) $R_W(\tau) = \delta(\tau)$ is the autocorrelation function whose Fourier transform is $S_W(f) = 1$.

(b) The output $Y(t)$ has power spectral density

$$S_Y(f) = |H(f)|^2 S_W(f) = |H(f)|^2 .$$

(c) Since $|H(f) = 1$ for $f \in [-\frac{1}{2}B, \frac{1}{2}B]$, the average power of $Y(t)$ is

$$\mathrm{E}[Y^2(t)] = \int_{-\infty}^{\infty} S_Y(f)\mathrm{d}f = \int_{-B/2}^{B/2} \mathrm{d}f = B$$

(d) Since the white noise $W(f)$ has zero mean, the expected value of the filter output is

$$\mathrm{E}[Y(t)] = \mathrm{E}[W(t)]H(0) = 0$$

Problem 8.7

A white Gaussian noise process $N(t)$ with power spectral density of 10^{-15} W/Hz is the input to a lowpass filter $H(f) = 10^3 e^{-10^{-6}|f|}$. Find the following properties of the output $Y(t)$:

(a) The expected value μ_Y

(b) The output power spectral density $S_Y(f)$

(c) The average power $\mathrm{E}[Y^2(t)]$

(d) $\mathrm{P}[Y(t) > 0.01]$

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(b) The output power spectral density $S_Y(f)$
(c) The average power $\mathrm{E}[Y^2(t)]$
(d) $\mathrm{P}[Y(t) > 0.01]$

(a) $\mathrm{E}[N(t)] = \mu_N = 0 \Rightarrow \mu_Y = \mu_N H(0) = 0$
(b) $S_Y(f) = |H(f)|^2 S_N(f) = 10^{-9} e^{-2\times 10^{-6}|f|}$
(c) $\mathrm{E}[Y^2(t)] = \int_{-\infty}^{\infty} S_Y(f)\mathrm{d}f = \int_{-\infty}^{\infty} 10^{-9} e^{-2\cdot 10^{-6}|f|}\mathrm{d}f =$

$$2 \cdot 10^{-9} \int_0^{\infty} e^{-2\cdot 10^{-6} f}\mathrm{d}f = 10^{-3} \text{ W}$$

Problem 8.7

(d) Since $N(t)$ is a Gaussian process, Theorem 3 says $Y(t)$ is a Gaussian process. Thus the random variable $Y(t)$ is Gaussian with

$$E[Y(t)] = 0\,, \qquad \mathrm{var}[Y(t)] = E[Y^2(t)] = 10^{-3}$$

Thus we can use Table 4.2 to calculate

$$\begin{aligned} P[Y(t) > 0.01] &= P\left[\frac{Y(t)}{\sqrt{\mathrm{var}[Y(t)]}} > \frac{0.01}{\sqrt{\mathrm{var}[Y(t)]}}\right] \\ &= 1 - \Phi\left(\frac{0.01}{\sqrt{0.001}}\right) \\ &= 1 - \Phi(0.32) = 0.3745 \end{aligned}$$

Problem 8.9

Let $M(t)$ be a WSS random process with average power $\mathrm{E}[M^2(t)] = q$ and power spectral density $S_M(f)$. The Hilbert transform of $M(t)$ is $\hat{M}(t)$, a signal obtained by passing $M(t)$ through a linear time-invariant filter with frequency response

$$H(f) = -j\,\mathrm{sgn}(f) = \begin{cases} -j & f \geq 0\,, \\ j & f < 0\,. \end{cases}$$

(a) Find the power spectral density $S_{\hat{M}}(f)$ and the average power $\hat{q} = \mathrm{E}[\hat{M}^2(t)]$.

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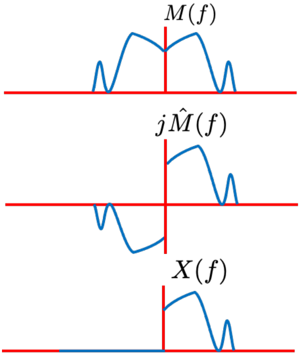

$x(t) = M(t) + j\hat{M}(t)$ is the "analytic signal": $X(f) = 0$ for $f < 0$

Problem 8.9 (cont'd)

(a) Note that $|H(f)| = 1$. This implies $S_{\hat{M}}(f) = S_M(f)$. Thus the average power of $\hat{M}(t)$ is

$$\hat{q} = \int_{-\infty}^{\infty} S_{\hat{M}}(f)\mathrm{d}f = \int_{-\infty}^{\infty} S_M(f)\mathrm{d}f = q$$

Problem 8.9 (cont'd)

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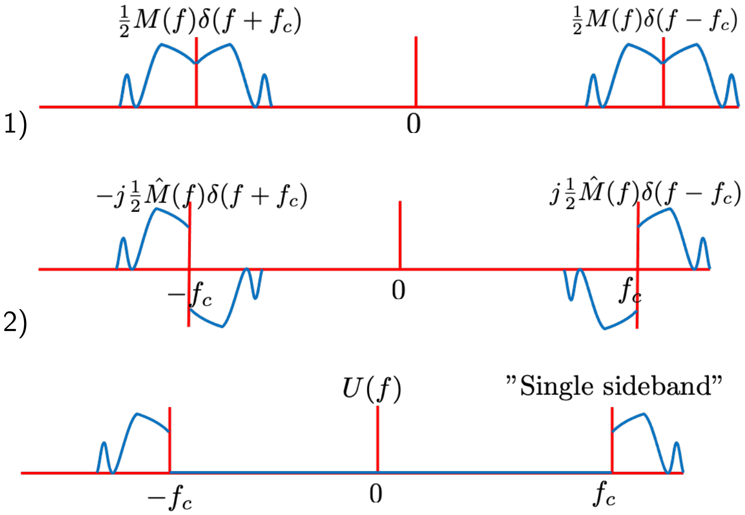
(b) In a single sideband communication system, the upper sideband signal is

$$U(t) = M(t)\cos(2\pi f_c t + \Theta) - \hat{M}(t)\sin(2\pi f_c t + \Theta)$$

where Θ has a uniform PDF over $[0, 2\pi)$, independent of $M(t)$ and $\hat{M}(t)$. What is the average power $\mathrm{E}[U^2(t)]$?

Problem 8.9 (cont'd) – Idea



$$\Rightarrow \mathsf{E}[U^2(t)] = q$$

Problem 8.9 (cont'd) – Idea

(b) The average power of the upper sideband signal is

$$\begin{aligned} \mathrm{E}[U^2(t)] &= \mathrm{E}[M^2(t)\cos^2(2\pi f_c t + \Theta)] \\ &\quad -\mathrm{E}\left[2M(t)\hat{M}(t)\cos(2\pi f_c t + \Theta)\sin(2\pi f_c t + \Theta)\right] \\ &\quad +\mathrm{E}\left[\hat{M}^2(t)\sin^2(2\pi f_c t + \Theta)\right] \end{aligned}$$

Use:

$$\begin{aligned} \mathrm{E}[\cos^2(2\pi f_c t + \Theta)] &= \frac{1}{2} \\ \mathrm{E}[\sin^2(2\pi f_c t + \Theta)] &= \frac{1}{2} \\ \mathrm{E}[2\sin(2\pi f_c t + \Theta)\cos(2\pi f_c t + \Theta)] &= \mathrm{E}[\sin(4\pi f_c t + 2\Theta)] = 0 \end{aligned}$$

Problem 8.9 (cont'd)

Since $M(t)$ and $\hat{M}(t)$ are independent of Θ, the average power of the upper sideband signal is

$$\begin{aligned} \mathrm{E}[U^2(t)] &= \mathrm{E}[M^2(t)]\,\mathrm{E}[\cos^2(2\pi f_c t + \Theta)] \\ &\quad -\mathrm{E}[M(t)\hat{M}(t)]\,\mathrm{E}[2\cos(2\pi f_c t + \Theta)\sin(2\pi f_c t + \Theta)] \\ &\quad +\mathrm{E}[\hat{M}^2(t)]\,\mathrm{E}[\sin^2(2\pi f_c t + \Theta)] \\ &= q/2 + 0 + q/2 \;=\; q \end{aligned}$$

That's all for now!

May the odds be ever in your favor...