EE2S31 Signal Processing – Stochastic Processes
Lecture 7: Power Spectral Density – Suppl. 5, 6

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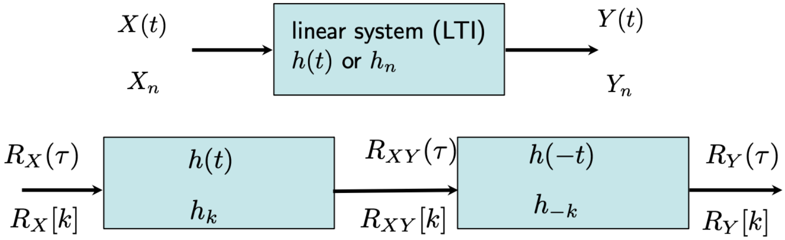
Summarizing

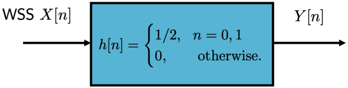


$$R_Y(\tau) = h(-\tau) * h(\tau) * R_X(\tau)$$

$$R_Y[k] = h_{-k} * h_k * R_X[k]$$

Example: FIR filter

$$\mathrm{E}[X_n] = 1\,, \qquad R_X[k] = \begin{cases} 4 & k = 0 \\ 2 & k = \pm 1 \\ 0 & |k| \geq 2 \end{cases}$$



Determine $\mathrm{E}[Y_n]$, $R_{XY}[k]$, $R_Y[k]$.

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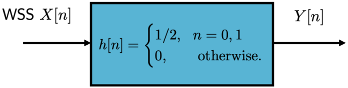


Determine $\mathrm{E}[Y_n]$, $R_{XY}[k]$, $R_Y[k]$.

- $\mathrm{E}[Y_n] = \mathrm{E}[X_n] \sum_n h_n = 1$
- Method 1 (only works for short FIR filters): $Y_n = \frac{1}{2} X_n + \frac{1}{2} X_{n-1}$

$$\begin{aligned} R_{XY}[k] &= \mathrm{E}[X_n Y_{n+k}] = \mathrm{E}\left[X_n \left(\frac{1}{2} X_{n+k} + \frac{1}{2} X_{n+k-1}\right)\right] \\ &= \frac{1}{2} R_X[k] + \frac{1}{2} R_X[k-1] \end{aligned}$$

- Method 2:

$$R_{XY}[k] = \sum_{j=-\infty}^{\infty} h_j R_X[k-j] = \frac{1}{2} R_X[k] + \frac{1}{2} R_X[k-1]$$

Example: FIR filter

- Method 1:

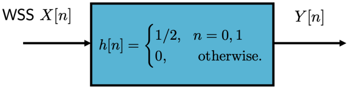
$$\begin{aligned} R_Y[k] &= \mathrm{E}[Y_n Y_{n+k}] = \mathrm{E}\left[\left(\frac{1}{2}X_n + \frac{1}{2}X_{n-1}\right)\left(\frac{1}{2}X_{n+k} + \frac{1}{2}X_{n+k-1}\right)\right] \\ &= \frac{1}{2}R_X[k] + \frac{1}{4}R_X[k-1] + \frac{1}{4}R_X[k+1] \end{aligned}$$

- Method 2: using $R_{XY}[k] = \frac{1}{2}R_X[k] + \frac{1}{2}R_X[k-1]$:

$$\begin{aligned} R_Y[k] &= \sum_{i=-\infty}^{\infty} h_{-i} R_{XY}[k-i] = h_0 R_{XY}[k] + h_1 R_{XY}[k+1] \\ &= \frac{1}{2}\left(\frac{1}{2}R_X[k] + \frac{1}{2}R_X[k-1]\right) + \frac{1}{2}\left(\frac{1}{2}R_X[k+1] + \frac{1}{2}R_X[k]\right) \\ &= \frac{1}{2}R_X[k] + \frac{1}{4}R_X[k-1] + \frac{1}{4}R_X[k+1] \end{aligned}$$

Example: FIR filter

Finally, using



$$\mathrm{E}[X_n] = 1\,, \qquad R_X[k] = \begin{cases} 4 & k = 0 \\ 2 & k = \pm 1 \\ 0 & |k| \geq 2 \end{cases}$$

we obtain

$$\begin{aligned} R_Y[k] &= \frac{1}{2} R_X[k] + \frac{1}{4} R_X[k-1] + \frac{1}{4} R_X[k+1] \\ &= \begin{cases} 3 & k = 0 \\ 2 & k = \pm 1 \\ 0.5 & k = \pm 2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

FIR and IIR filters

Notice that this was a simple example of an FIR filter. More in general:

- Finite Impulse Response (FIR) filter:

$$Y_n = \sum_{i=0}^{M} h_i X_{n-i}$$

- Infinite Impulse Response (IIR) filter:

$$Y_n = \sum_{i=0}^{\infty} h_i X_{n-i} = \sum_{j=1}^{N} a_j Y_{n-j} + \sum_{i=0}^{M} b_i X_{n-i}$$

FIR and IIR filters

Terminology

If we input white noise into these LTI systems, then

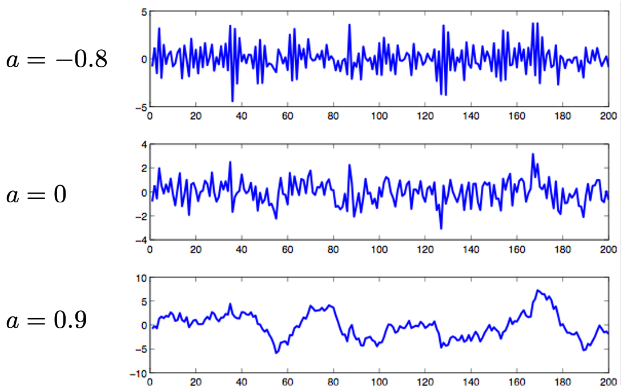
- the output of an FIR filter is called a moving average (MA) process,
- the output of an IIR filter is called an autoregressive (AR) process,
- a combination of the two is called an ARMA process.

Applications of AR models:

- Biomedical signals: EEG signals, heart rhythm variability
- Many speech processing systems (including speech compression in GSM)

Example of AR(1) realizations

Last week, we saw a first order AR process: $Y_n = aY_{n-1} + X_n$, with X_n: white Gaussian noise with variance σ^2



$$R_Y[k] = \frac{\sigma^2}{1 - a^2} a^{|k|}$$

Modeling biomedical signals using AR models

EEG signal examples

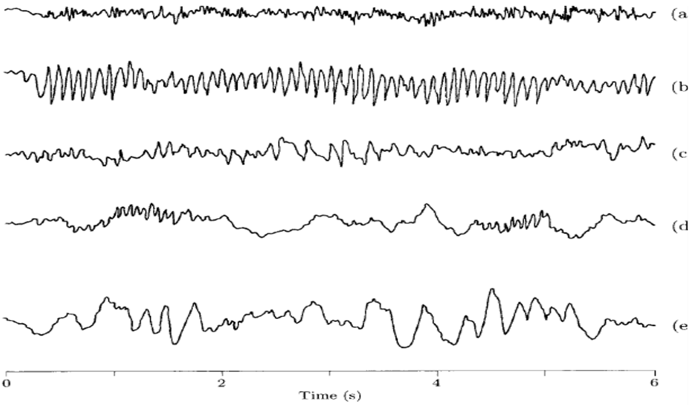


Figure 2.3: Electroencephalographic rhythms observed during various states fro wakefulness to sleep: (a) excited, (b) relaxed, (c) drowsy, (d) asleep, and (e) deep asleep. This example is classical and was originally presented by the famous EE pioneer H.H. Jasper [12].

Modeling biomedical signals using AR models

EEG signal at the onset of an epileptic seizure

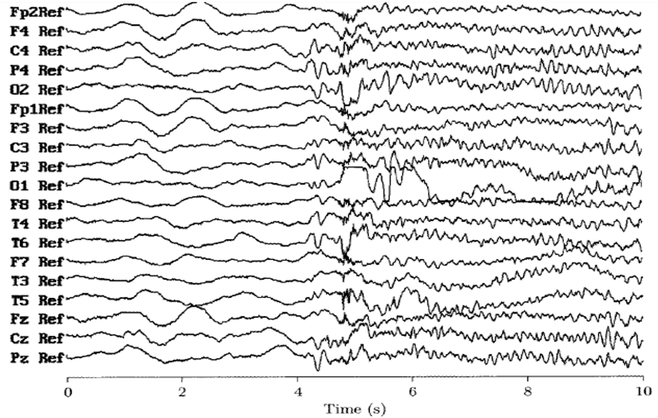
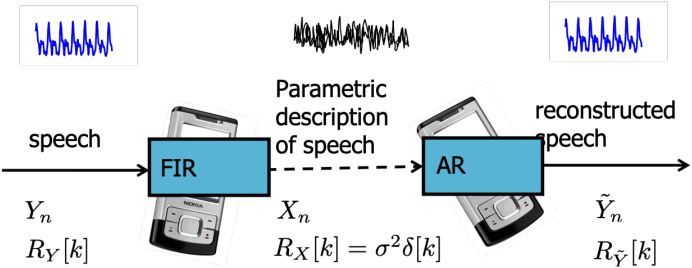


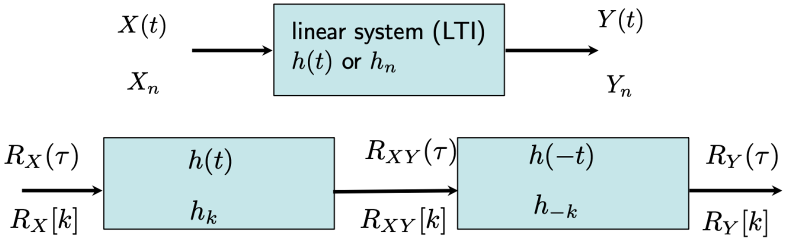
Figure 2.9: A multichannel EEG showing the onset of a primary generalized seizure about halfway into the recording. (Reprinted from Wong [16] with permis-

Speech modeling



- **Analysis:** find coefficients $\{a_k\}$ of an FIR filter such that $Y_n - a_1 Y_{n-1} - \cdots - a_p Y_{n-p} = X_n$ is “white”
- Transmit filter coefficients $\{a_k\}$, σ^2
- **Synthesis:** generate white noise X_n and use $\{a_k\}$ as AR filter to reconstruct $Y_n = a_1 Y_{n-1} + \cdots + a_p Y_{n-p} + X_n$.

Suppl. 5, 6: Power spectral density



$R_X[k]$ is a *deterministic* sequence that captures properties of the RV X_n.

What about the DTFT of X_n?

Notation

Notation issues, compared to EE2S11 Signals & Systems:

- Using f for continuous-time frequency (not F)
- Using f [Hz] instead of Ω [rad/s]; **table 1, 2** is a bit different ($\Omega \Rightarrow 2\pi f$)
- Using $X(f)$ and $X(t)$, or $x(t)$: confusion between FT and RV

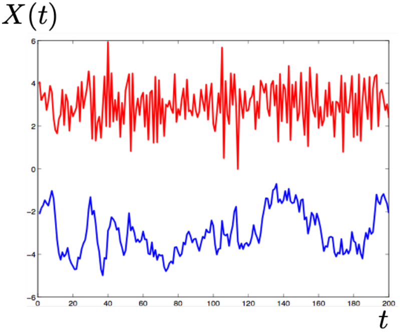
Regarding Section 6:

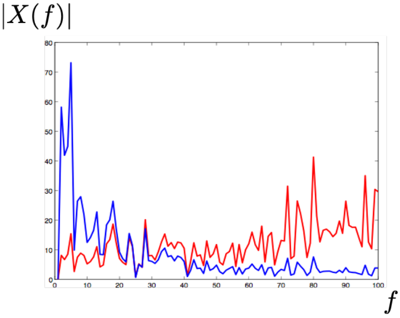
- Using ϕ for discrete-time normalized frequency (not f)
- Using ϕ (normalized frequency) instead of ω; **table 3** is a bit different ($\omega \Rightarrow 2\pi\phi$)
- Mixing use of x_n and X_n, and h_n or $h[n]$

The Fourier transform

Fourier transforms are commonly used in signal processing to describe

- Frequency content of deterministic signals
- Frequency characteristics of filters

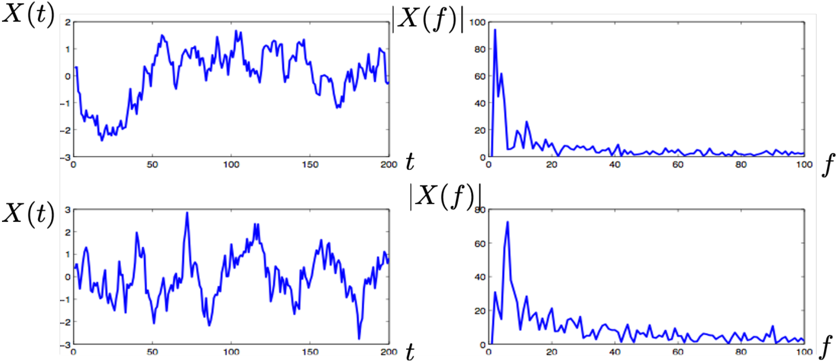




- The FT of a random signal is another random signal!

Fourier Transform for random signals?

For random signals, the Fourier transform of a single realization is less informative as the result depends on the particular realization.



- Solution: average! I.e. consider $\mathrm{E}[|X(f)|^2]$
  (Question: why not $\mathrm{E}[X(f)]$?)

Power Spectral Density (PSD)

Given $X(t)$, consider a rectangular windowed version

$$X_T(t) = \begin{cases} X(t) & -T \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

The power spectral density of a WSS process $X(t)$ can be defined as:

$$S_X(f) = \lim_{T\to\infty} \frac{1}{2T} \mathrm{E}\left[|X_T(f)|^2\right] = \lim_{T\to\infty} \frac{1}{2T} \mathrm{E}\left[\left|\int_{-T}^{T} X(t) e^{-j2\pi f t} \mathrm{d}t\right|^2\right]$$

- Take Fourier transform of RV $X_T(t)$
- Take absolute value (amplitude spectrum), squared
- Average (expected value), normalize by $2T$

The windowing by T is needed because WSS processes run forever (infinite energy) $\Rightarrow$ convert to power by $1/(2T)$

Wiener-Khintchine theorem

$$
\begin{aligned}
\mathrm{E}\left[|X_T(f)|^2\right] &= \mathrm{E}\left[\left(\int_{-T}^{T} X(t)e^{-j2\pi ft}\mathrm{d}t\right)\left(\int_{-T}^{T} X(t')e^{j2\pi ft'}\mathrm{d}t'\right)\right] \\
&= \int_{-T}^{T}\int_{-T}^{T} \underbrace{\mathrm{E}\left[X(t)X(t')\right]}_{R_X(t-t')} e^{-j2\pi f(t-t')}\mathrm{d}t\mathrm{d}t'
\end{aligned}
$$

$$
S_X(f) = \lim_{T\to\infty}\frac{1}{2T}\mathrm{E}\left[|X_T(f)|^2\right] = \cdots = \int_{-\infty}^{\infty} R_X(\tau)e^{-j2\pi f\tau}\mathrm{d}\tau
$$

Compute the average power at a frequency, for (in principle) infinitely long signals.

Wiener-Khintchine

For time-continuous WSS random processes $X(t)$, the power spectral density (PSD) is the Fourier transform of $R_X(\tau)$:

$$
\begin{aligned}
S_X(f) &= \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f \tau} \mathrm{d}\tau \\
R_X(\tau) &= \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f \tau} \mathrm{d}f
\end{aligned}
$$

- $S_X(f) \geq 0$ for all f
- $\int_{-\infty}^{\infty} S_X(f) \mathrm{d}f = R_X(0) = \mathrm{E}[X^2(t)]$
- $S_X(-f) = S_X(f)$
- Note: no $\frac{1}{2\pi}$ in the IFT as we work with f, not Ω.

Interpretations

- The PSD of a random process/signal gives the average power of the signal as function of frequency. Since f is continuous, it is a density.
- The PSD can be calculated from the autocorrelation function of the WSS random process.

If $Y(t)$ is the output of a filter $h(t)$ with input $X(t)$, we saw

$$R_Y(\tau) = h(\tau) * h(-\tau) * R_X(\tau)$$

$$\Rightarrow \quad S_Y(f) = H(f) \cdot H^*(f) \cdot S_X(f) = |H(f)|^2 \, S_X(f)$$

This shows how the filter modifies frequency components of $X(t)$ individually. ($\Rightarrow$ Supplement Sections 7, 8; next lecture)

Fourier transform pairs

Table 1 (p.29)

Time function	Fourier Transform
$\delta(\tau)$	1
1	$\delta(f)$
$\delta(\tau-\tau_0)$	$e^{-j2\pi f\tau_0}$
$u(\tau)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$e^{j2\pi f_0\tau}$	$\delta(f-f_0)$
$\cos 2\pi f_0\tau$	$\frac{1}{2}\delta(f-f_0)+\frac{1}{2}\delta(f+f_0)$
$\sin 2\pi f_0\tau$	$\frac{1}{2j}\delta(f-f_0)-\frac{1}{2j}\delta(f+f_0)$
$ae^{-a\tau}u(\tau)$	$\frac{a}{a+j2\pi f}$
$ae^{-a\|\tau\|}$	$\frac{2a^2}{a^2+(2\pi f)^2}$
$ae^{-\pi a^2\tau^2}$	$e^{-\pi f^2/a^2}$
$\mathrm{rect}(\tau/T)$	$T\,\mathrm{sinc}(fT)$
$\mathrm{sinc}(2W\tau)$	$\frac{1}{2W}\mathrm{rect}(\frac{f}{2W})$

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

$$\mathrm{rect}(x) = \begin{cases} 1 & |x| < \frac{1}{2} \\ 0 & \text{o.w.} \end{cases}$$

Fourier transform properties

Table 2 (p.30)

Time function	Fourier Transform
$g(\tau - \tau_0)$	$G(f)e^{-j2\pi f\tau_0}$
$g(\tau)e^{j2\pi f_0\tau}$	$G(f - f_0)$
$g(-\tau)$	$G^*(f)$
$\frac{dg(\tau)}{d\tau}$	$j2\pi fG(f)$
$\int_{-\infty}^{\tau} g(v)\,dv$	$\frac{G(f)}{j2\pi f} + \frac{G(0)}{2}\delta(f)$
$\int_{-\infty}^{\infty} h(v)g(\tau - v)\,dv$	$G(f)H(f)$
$g(t)h(t)$	$\int_{-\infty}^{\infty} H(f')G(f - f')\,df'$

Example 1

Let $X(t)$ be white noise: WSS, zero mean, with $R_X(\tau) = N_0\delta(\tau)$.

Determine $S_X(f)$.

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Let $X(t)$ be white noise: WSS, zero mean, with $R_X(\tau) = N_0\delta(\tau)$.

Determine $S_X(f)$.

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)e^{-j2\pi f\tau}\mathrm{d}\tau = N_0 \int_{-\infty}^{\infty} \delta(\tau)e^{-j2\pi f\tau}\mathrm{d}\tau = N_0$$

The PSD is flat, all frequencies are equally strong, hence "white" (cf. white light)

Example 2

Given is a WSS process $X(t)$ with

$$R_X(\tau) = Ae^{-b|\tau|}, \quad b > 0.$$

Derive the PSD $S_X(f)$ and calculate the average power $\mathrm{E}[X^2(t)]$.

Example 2

Given is a WSS process $X(t)$ with

$$R_X(\tau) = Ae^{-b|\tau|}, \quad b > 0.$$

Derive the PSD $S_X(f)$ and calculate the average power $\mathrm{E}[X^2(t)]$.

Using Table 1 (Fourier transform pairs), we learn that

$$ae^{-a|\tau|} \Leftrightarrow \frac{2a^2}{a^2 + (2\pi f)^2}$$

$$S_X(f) = \frac{A}{b}\frac{2b^2}{b^2 + (2\pi f)^2} = \frac{2Ab}{b^2 + (2\pi f)^2}$$

The average power is:

$$\begin{aligned} \mathrm{E}[X^2(t)] &= R_X(0) = A \\ &= \int_{-\infty}^{\infty} S_X(f)\mathrm{d}f = \left[\frac{A}{\pi}\arctan(2\pi f/b)\right]_{-\infty}^{\infty} \end{aligned}$$

Problem 5.1

$X(t)$ is a wide sense stationary process with autocorrelation function

$$R_X(\tau) = 10\frac{\sin(2000\pi\tau) + \sin(1000\pi\tau)}{2000\pi\tau}$$

What is the power spectral density of $X(t)$?

Problem 5.1

$X(t)$ is a wide sense stationary process with autocorrelation function

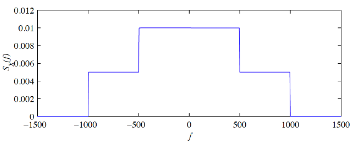
$$R_X(\tau) = 10\frac{\sin(2000\pi\tau) + \sin(1000\pi\tau)}{2000\pi\tau}$$

What is the power spectral density of $X(t)$?

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

In terms of the $\text{sinc}(\cdot)$ function, we obtain

$$R_X(\tau) = 10\,\text{sinc}(2000\tau) + 5\,\text{sinc}(1000\tau)\,.$$



From Table 1,

$$S_X(f) = \frac{10}{2000}\text{rect}\left(\frac{f}{2000}\right) + \frac{5}{1000}\text{rect}\left(\frac{f}{1000}\right)$$

Problem 5.2

$X(t)$ is a wide sense stationary process with $\mu_X = 0$, and $Y(t) = X(\alpha t)$ where α is a nonzero constant. Find $R_Y(\tau)$ in terms of $R_X(\tau)$. Is $Y(t)$ wide sense stationary? If so, find the power spectral density $S_Y(f)$.

Problem 5.2

$X(t)$ is a wide sense stationary process with $\mu_X = 0$, and $Y(t) = X(\alpha t)$ where α is a nonzero constant. Find $R_Y(\tau)$ in terms of $R_X(\tau)$. Is $Y(t)$ wide sense stationary? If so, find the power spectral density $S_Y(f)$.

The process $Y(t)$ has expected value $\mathrm{E}[Y(t)] = 0$. The autocorrelation function of $Y(t)$ is

$$\begin{aligned} R_Y(t,\tau) &= \mathrm{E}[Y(t)Y(t+\tau)] \\ &= \mathrm{E}[X(\alpha t)X(\alpha(t+\tau))] = R_X(\alpha\tau)\,. \end{aligned}$$

Thus, $Y(t)$ is WSS. The power spectral density is

$$S_Y(f) = \int_{-\infty}^{\infty} R_X(\alpha\tau) e^{-j2\pi f\tau} \mathrm{d}\tau\,.$$

Problem 5.2 (cont'd)

- For $\alpha > 0$, the substitution $\tau' = \alpha\tau$ gives

$$S_Y(f) = \frac{1}{\alpha}\int_{-\infty}^{\infty} R_X(\tau')e^{-j2\pi(f/\alpha)\tau'}\mathrm{d}\tau' = \frac{1}{\alpha}S_X(f/\alpha)$$

- For $\alpha < 0$, the substitution $\tau' = -\alpha\tau$ gives

$$\begin{aligned} S_Y(f) &= \frac{1}{-\alpha}\int_{-\infty}^{\infty} R_X(-\tau')e^{-j2\pi(-f/\alpha)\tau'}\mathrm{d}\tau' \\ &= \frac{1}{-\alpha}S_X(-f/\alpha) \qquad \text{using } R_X(-\tau') = R_X(\tau') \end{aligned}$$

Altogether,

$$S_Y(f) = \frac{1}{|\alpha|}S_X\left(\frac{f}{\alpha}\right)$$

PSD for Discrete-time stochastic processes

For a discrete-time stochastic WSS process X_n:

$$
\begin{aligned}
S_X(\phi) &= \sum_{k-\infty}^{\infty} R_X[k] e^{-j2\pi\phi k} \\
R_X[k] &= \int_{-1/2}^{1/2} S_X(\phi) e^{j2\pi\phi k} \mathrm{d}\phi
\end{aligned}
$$

- $S_X(\phi) \geq 0$ for all ϕ
- $\int_{-1/2}^{1/2} S_X(\phi)\mathrm{d}\phi = R_X[0] = \mathrm{E}[X_n^2]$
- $S_X(-\phi) = S_X(\phi)$
- For any integer n: $S_X(\phi + n) = S_X(\phi)$ (spectrum is periodic.)

Discrete-time Fourier transform pairs

Table 3 (p.37)

Discrete Time function	Discrete Time Fourier Transform
$\delta[n] = \delta_n$	1
1	$\delta(\phi)$
$\delta[n-n_0] = \delta_{n-n_0}$	$e^{-j2\pi\phi n_0}$
$u[n]$	$\dfrac{1}{1-e^{-j2\pi\phi}} + \dfrac{1}{2}\sum_{k=-\infty}^{\infty} \delta(\phi+k)$
$e^{j2\pi\phi_0 n}$	$\sum_{k=-\infty}^{\infty} \delta(\phi-\phi_0-k)$
$\cos 2\pi\phi_0 n$	$\frac{1}{2}\delta(\phi-\phi_0) + \frac{1}{2}\delta(\phi+\phi_0)$
$\sin 2\pi\phi_0 n$	$\frac{1}{2j}\delta(\phi-\phi_0) - \frac{1}{2j}\delta(\phi+\phi_0)$
$a^n u[n]$	$\dfrac{1}{1-ae^{-j2\pi\phi}}$
$a^{\lvert n\rvert}$	$\dfrac{1-a^2}{1+a^2-2a\cos 2\pi\phi}$
g_{n-n_0}	$G(\phi)e^{-j2\pi\phi n_0}$
$g_n e^{j2\pi\phi_0 n}$	$G(\phi-\phi_0)$
g_{-n}	$G^*(\phi)$
$\sum_{k=-\infty}^{\infty} h_k g_{n-k}$	$G(\phi)H(\phi)$
$g_n h_n$	$\int_{-1/2}^{1/2} H(\phi')G(\phi-\phi')\,d\phi'$

Problem 6.1

X_n is a wide sense stationary discrete-time random sequence with autocorrelation sequence $R_X[k]$ such that

$$R_X[k] = \delta[k] + (0.1)^{|k|}, \qquad k = 0, \pm 1, \pm 2, \cdots$$

Find the power spectral density $S_X(\phi)$.

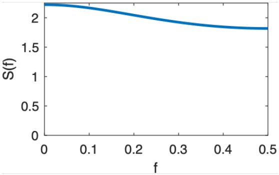
Problem 6.1

X_n is a wide sense stationary discrete-time random sequence with autocorrelation sequence $R_X[k]$ such that

$$R_X[k] = \delta[k] + (0.1)^{|k|}, \qquad k = 0, \pm 1, \pm 2, \cdots$$

Find the power spectral density $S_X(\phi)$.

We can find the PSD directly from Table 3 with $(0.1)^{|k|}$, corresponding to $a^{|k|}$. The table yields



$$\begin{aligned} S_X(\phi) &= 1 + \frac{1-(0.1)^2}{1+(0.1)^2 - 2(0.1)\cos 2\pi\phi} \\ &= \frac{2 - 0.2\cos 2\pi\phi}{1.01 - 0.2\cos 2\pi\phi} \end{aligned}$$

Example

Given a zero-mean WSS process X_n with $R_X[k]$

$$R_X[k] = \begin{cases} \sigma^2(2 - |k|)/4 & k = -1, 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the PSD of X_n?

Example

Given a zero-mean WSS process X_n with $R_X[k]$

$$R_X[k] = \begin{cases} \sigma^2(2-|k|)/4 & k=-1,0,1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the PSD of X_n?

$$\begin{aligned} S_X(\phi) &= \sum_{n=-1}^{1} R_X[k]e^{-j2\pi n\phi} = \sigma^2\left[\frac{2-1}{4}e^{j2\pi\phi} + \frac{2}{4} + \frac{2-1}{4}e^{-j2\pi\phi}\right] \\ &= \frac{\sigma^2}{2} + \frac{\sigma^2}{2}\cos(2\pi\phi) \end{aligned}$$

Problem

Given the power spectral density of a WSS sequence X_n:

$$S_X(\phi) = 5 + 4\cos(2\pi\phi)$$

Find the corresponding autocorrelation sequence $R_X[k]$. Calculate the average power of X_n.

Problem

Given the power spectral density of a WSS sequence X_n:

$$S_X(\phi) = 5 + 4\cos(2\pi\phi)$$

Find the corresponding autocorrelation sequence $R_X[k]$. Calculate the average power of X_n.

$$S_X(\phi) = 5 + 4\cos(2\pi\phi) = 5 + 2e^{j2\pi\phi} + 2e^{-j2\pi\phi}$$

Table 3 shows: $R_X[k] = 5\delta[k] + 2\delta[k-1] + 2\delta[k+1]$.

The average power of X_n is $\mathrm{E}[X_n^2] = R_X[0] = 5$.

To do:

- Study Sections 5 and 6
- Check old exams for related exercises

Next lecture, we'll finish the course with Supplement Sections 7 and 8.