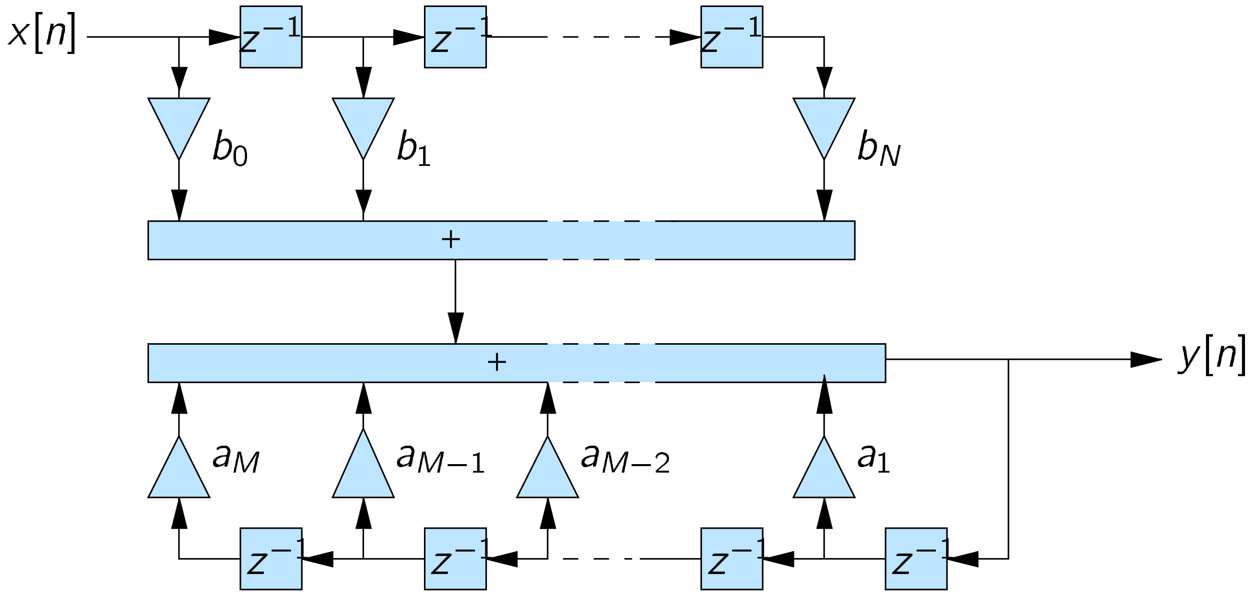
Filter realizations: lattice filter structures

- Direct form realizations (recap)
- Realization of allpass filters (Schur recursion)
- Lattice filters

Recursive filter: direct form no. 1

Realization of a general difference equation ($a_0 = 1$)

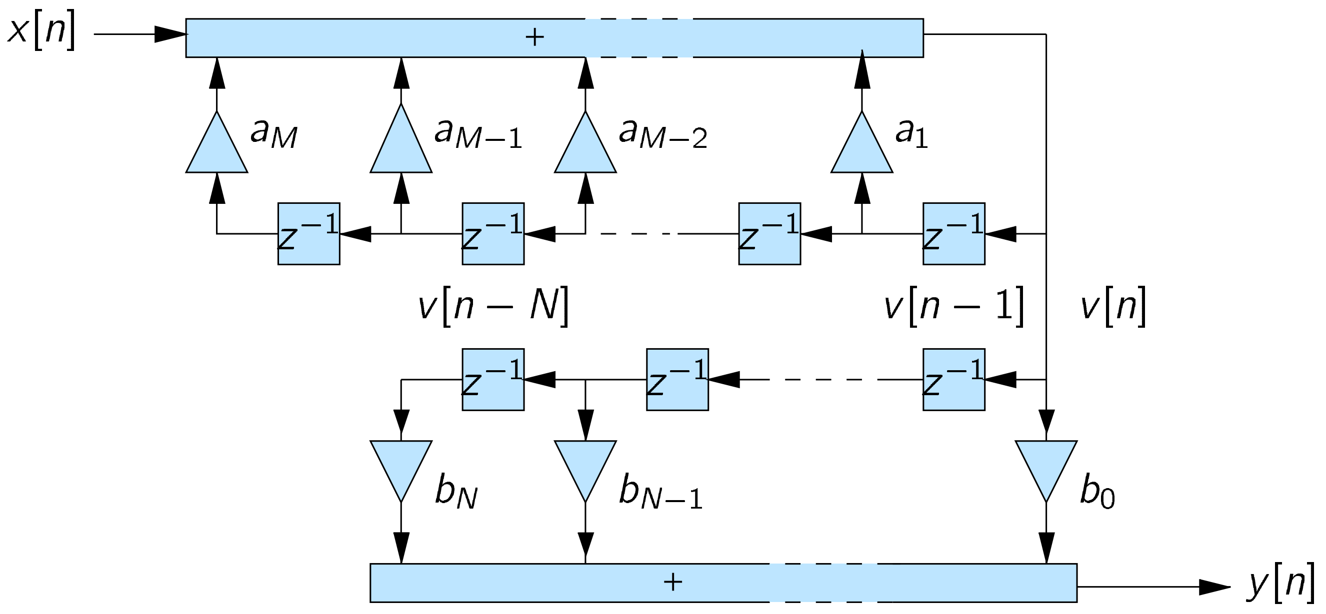
$$y[n] = \sum_{i=0}^{N} b_i x[n-i] + \sum_{i=1}^{M} a_i y[n-i] \quad \Rightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{N} b_i z^{-i}}{1 - \sum_{i=1}^{M} a_i z^{-i}} = \frac{B(z)}{A(z)}$$



This is not a minimal structure ($M + N$ delays instead of $\max(M, N)$ delays).

Recursive filter: direct form no. 2

Use the commutative property of the convolution: $h_1 * h_2 = h_2 * h_1$. The two parts of the system can be swapped.

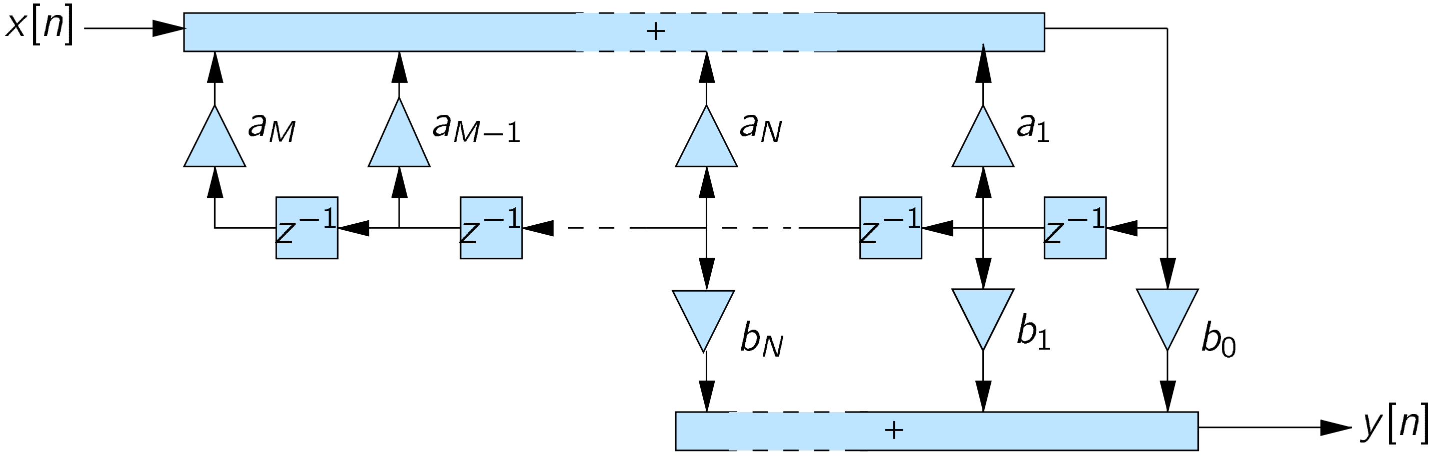


Notice that the delay lines can be merged (they transport the same signal $v[n]$)

Discrete time filter structures: direct form

Recursive filter: direct form no. 2 (cont'd)

The resulting filter (minimal and canonical):



- The filter coefficients are directly related to the parameters of the difference equation.

Discrete time filter structures

Problems with the direct realization

- Very sensitive to small disturbances (quantization) of the coefficients.
- Hence, selective filters (poles close to the unit circle) easily become unstable.
- Hence, only used in a cascade of 2nd order sections.
- If coarsely quantized, only a few locations for poles close to the unit circle

We will investigate an alternative. It is based on a recursive realization of allpass filters: not directly but as a 'lattice'. This is used in selective filters (e.g. frequency band selectors in mobile phones), and in analog form as microwave strip filters.

We will use the Schur recursion (strongly connected to Delft. . .)

Allpass functions (recap)

$H(z)$ is an allpass function if $|H(e^{j\omega})| = 1$ for all ω.

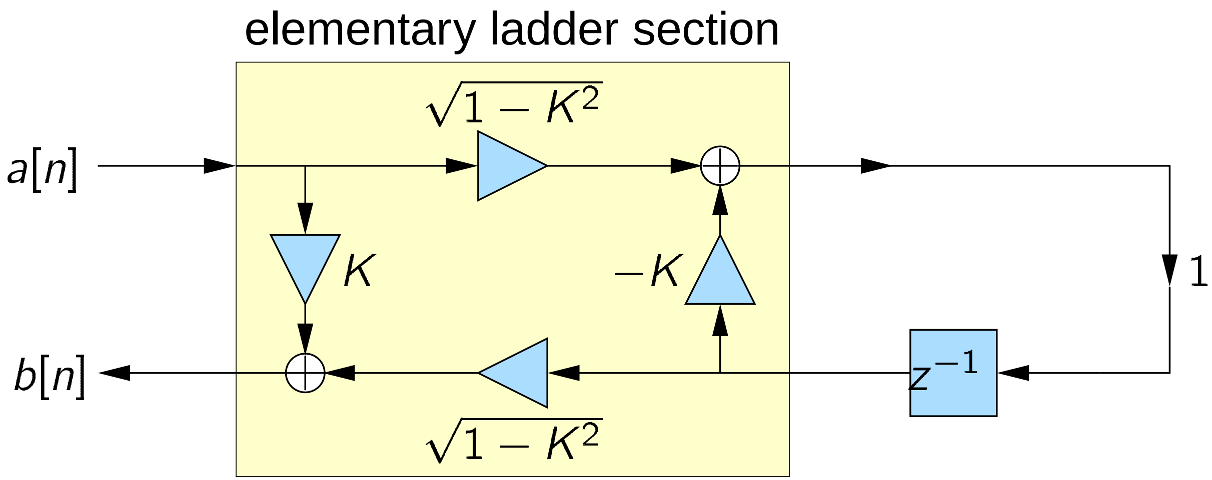
Example: $H(z) = z^{-1}$.

Proposition: every rational allpass function with real coefficients is of the form

$$H(z) = \frac{a_M + a_{M-1}z^{-1} + \cdots + a_1 z^{-M+1} + z^{-M}}{1 + a_1 z^{-1} + \cdots + a_{M-1} z^{-M+1} + a_M z^{-M}} = \frac{z^{-M}A(z^{-1})}{A(z)} =: \frac{\widetilde{A}(z)}{A(z)}$$

Hence, the numerator polynomial is the reverse of the denominator. (For complex coefficients, the numerator coefficients also should be conjugated.)

Allpass filter realization—first order



The response is (for $K \leq 1$)

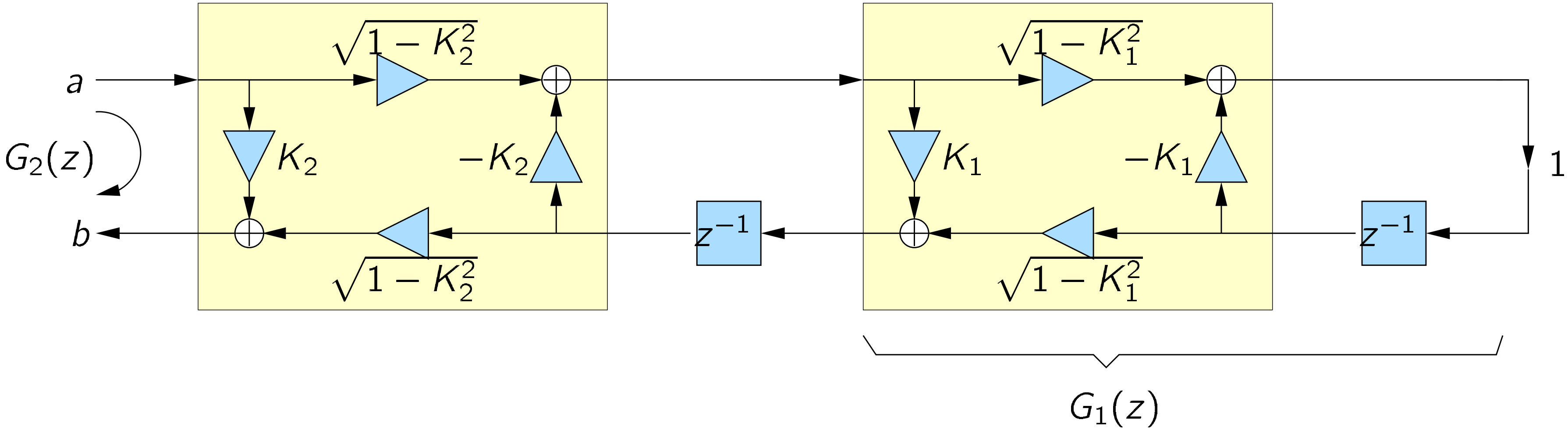
$$G(z) = \frac{B(z)}{A(z)} = K + (1 - K^2)\frac{z^{-1}}{1 + Kz^{-1}} = \frac{K + z^{-1}}{1 + Kz^{-1}}$$

This is the general form of a first order allpass function.

K is called a reflection coefficient ($|K| \leq 1$)

Proposition: every (stable causal) rational allpass function can be realized by a cascade of such sections.

Allpass filter realization—2nd order

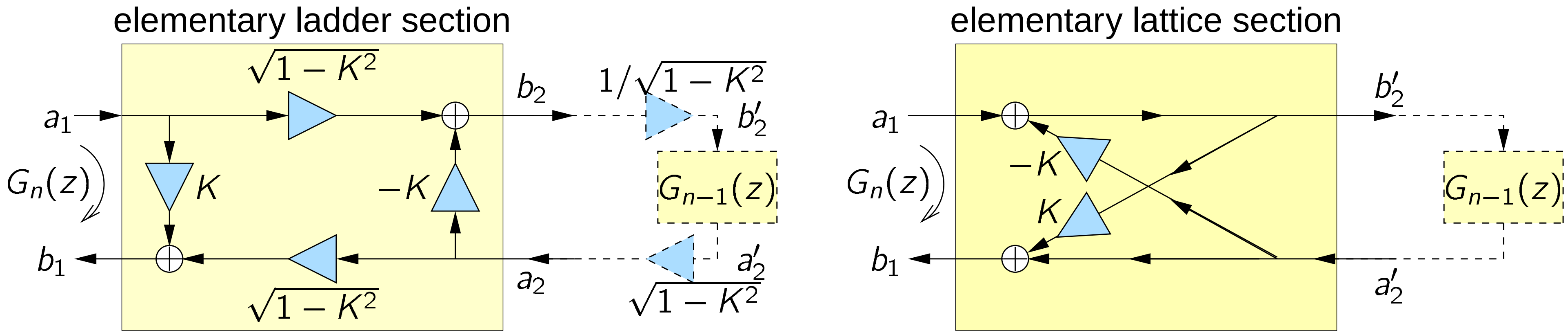


$$G_1(z) = \frac{K_1 + z^{-1}}{1 + K_1 z^{-1}}$$

$$\begin{aligned} G_2(z) &= \frac{K_2 + G_1(z)z^{-1}}{1 + K_2 G_1(z) z^{-1}} \\ &= \frac{K_2 + \frac{K_1 + z^{-1}}{1 + K_1 z^{-1}} z^{-1}}{1 + K_2 \frac{K_1 + z^{-1}}{1 + K_1 z^{-1}} z^{-1}} \\ &= \frac{K_2 + K_1(1+K_2)z^{-1} + z^{-2}}{1 + K_1(1+K_2)z^{-1} + K_2 z^{-2}} \end{aligned}$$

Discrete time filter structures

Allpass filter realization—lattice section



The elementary section is an orthogonal matrix (rotation):

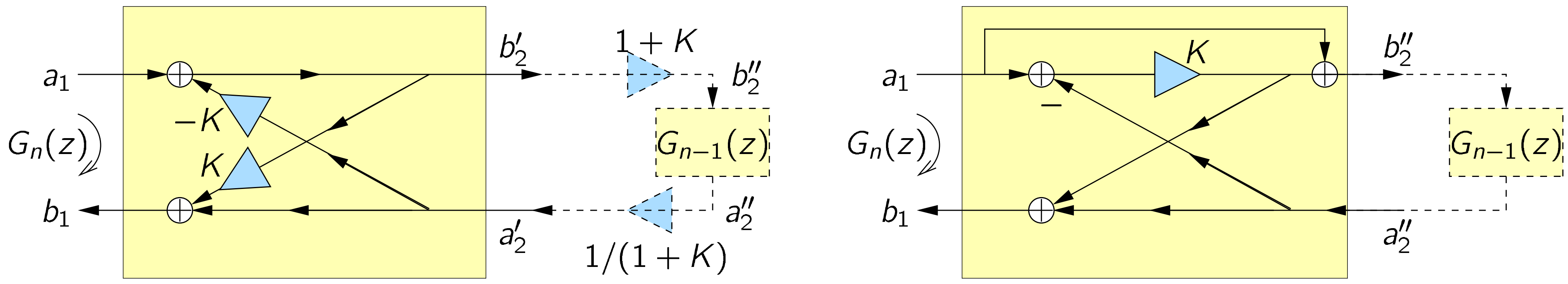
$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} K & \sqrt{1-K^2} \\ \sqrt{1-K^2} & -K \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

The closed-loop response does not change if we divide b_2 by $\sqrt{1-K^2}$ and multiply a_2 with it. This results in

$$\begin{bmatrix} b_1 \\ b_2' \end{bmatrix} = \begin{bmatrix} 1 & \\ & 1/\sqrt{1-K^2} \end{bmatrix} \begin{bmatrix} K & \sqrt{1-K^2} \\ \sqrt{1-K^2} & -K \end{bmatrix} \begin{bmatrix} 1 & \\ & \sqrt{1-K^2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2' \end{bmatrix} = \begin{bmatrix} K & 1-K^2 \\ 1 & -K \end{bmatrix} \begin{bmatrix} a_1 \\ a_2' \end{bmatrix}$$

Discrete time filter structures

Allpass filter realization—section using 1 multiplier



The number of multiplications can be further reduced by the following transform (the closed-loop response does not change)

$$\begin{bmatrix} b_1 \\ b_2'' \end{bmatrix} = \begin{bmatrix} 1 & \\ & 1+K \end{bmatrix} \begin{bmatrix} K & 1-K^2 \\ 1 & -K \end{bmatrix} \begin{bmatrix} 1 & \\ & 1/(1+K) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2'' \end{bmatrix} = \begin{bmatrix} K & 1-K \\ 1+K & -K \end{bmatrix} \begin{bmatrix} a_1 \\ a_2'' \end{bmatrix}$$

Additional advantage: the response is determined by only 1 parameter, and will be allpass even with quantization. This is a canonical section.

Discrete time filter structures

Realisation of a stable causal all-pass filter

Given

$$H(z) = \frac{B(z)}{A(z)} = \frac{a_N + a_{N-1}z^{-1} + \cdots + z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$H(z)$ is causal stable if all poles of $H(z)$ (the roots of $A(z)$) are inside the unit circle.

Note:

$$A(z) = (1 - p_1 z^{-1})(1 - p_2 z^{-1}) \cdots (1 - p_N z^{-1}) = 1 + a_1 z^{-1} + \cdots + a_N z^{-N}$$

with $a_N = p_1 p_2 \cdots p_N$. Necessary condition for stability: $|a_N| < 1$.

Discrete time filter structures

Realization of a stable causal all-pass filter using a lattice filter

Initialisation: Let $A_N(z) = A(z) = 1 + a_{N,1}z^{-1} + \cdots + a_{N,N}z^{-N}$.

We consider the allpass function

$$G_N(z) = \frac{z^{-N}A_N(z^{-1})}{A_N(z)} =: \frac{\widetilde{A}_N(z)}{A_N(z)} = \frac{a_{N,N} + a_{N,N-1}z^{-1} + \cdots + z^{-N}}{1 + a_{N,1}z^{-1} + \cdots + a_{N,N}z^{-N}}$$

Recursion: for $n = N, N-1, \cdots, 0$,

- Let $K_n = a_{n,n}$. If $|K_n| \geq 1$, then $A_n(z)$ is not stable (and the recursion stops)
- Reduce the degree:

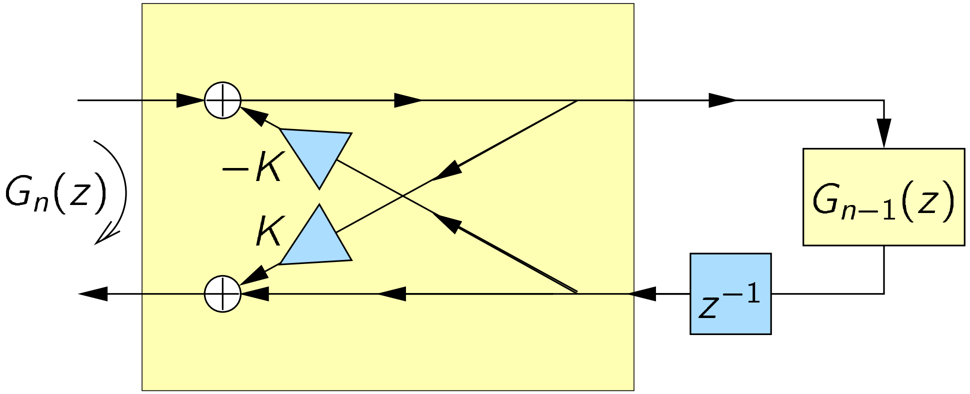
$$G_{n-1}(z) = z\,\frac{G_n(z) - K_n}{1 - K_n G_n(z)} = z\,\frac{\widetilde{A}_n(z) - K_n A_n(z)}{A_n(z) - K_n \widetilde{A}_n(z)} =: \frac{\widetilde{A}_{n-1}(z)}{A_{n-1}(z)}$$

The degree is reduced because $G_n(z) - K_n$ has a zero at $z = \infty$ which is canceled by the multiplication with z. Moreover, $G_{n-1}(z)$ is an allpass.

If the recursion does not stop prematurely, then $H(z)$ is stable.

Discrete time filter structures

Lattice recursion–derivation



$$G_n(z) = \frac{K_n + G_{n-1}z^{-1}}{1 + K_n G_{n-1} z^{-1}}$$

$$\Leftrightarrow \quad G_n(1 + K_n G_{n-1} z^{-1}) = K_n + G_{n-1} z^{-1}$$

$$\Leftrightarrow \quad G_{n-1}(G_n K_n z^{-1} - z^{-1}) = K_n - G_n$$

$$\Leftrightarrow \quad G_{n-1}(z) = z \frac{G_n(z) - K_n}{1 - K_n G_n(z)}$$

$G_n(z)$ only is a stable allpass if $|K_n| < 1$ and $G_{n-1}(z)$ is a stable allpass.

Example

Derive a lattice realization of the allpass filter $H(z) = \dfrac{0.3 + 0.8z^{-1} + 0.9z^{-2} + z^{-3}}{1 + 0.9z^{-1} + 0.8z^{-2} + 0.3z^{-3}}$

Recursion:

$$G_3(z) = \frac{\widetilde{A}_3(z)}{A_3(z)} := \frac{0.3 + 0.8z^{-1} + 0.9z^{-2} + z^{-3}}{1 + 0.9z^{-1} + 0.8z^{-2} + 0.3z^{-3}}$$

- $K_3 := a_3(3) = 0.3$. Note: $|K_3| < 1$, necessary for stability

$$G_2(z) = z\frac{\widetilde{A}_3(z) - 0.3A_3(z)}{A_3(z) - 0.3\widetilde{A}_3(z)} = \frac{0.5824 + 0.7253z^{-1} + z^{-2}}{1 + 0.7253z^{-1} + 0.5824z^{-2}} =: \frac{\widetilde{A}_2(z)}{A_2(z)}$$

- $K_2 = a_2(2) = 0.5824 < 1$: OK

$$G_1(z) = z\frac{\widetilde{A}_2(z) - 0.5824A_2(z)}{A_2(z) - 0.5824\widetilde{A}_2(z)} = \frac{0.45836 + z^{-1}}{1 + 0.45836z^{-1}} =: \frac{\widetilde{A}_1(z)}{A_1(z)}$$

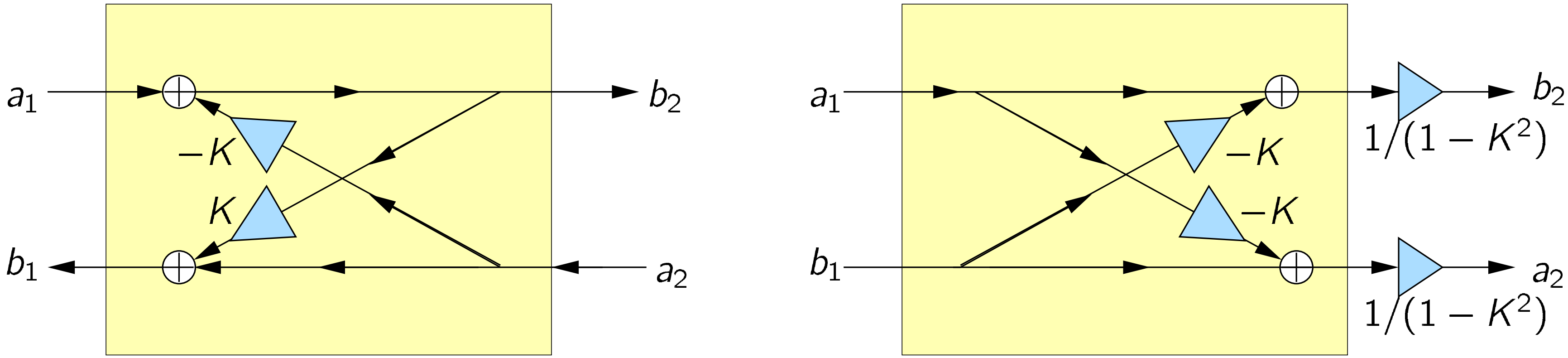
- $K_1 = a_1(1) = 0.45836 < 1$: OK

$$G_0(z) = z\frac{\widetilde{A}_1(z) - 0.45836A_1(z)}{A_1(z) - 0.45836\widetilde{A}_1(z)} = 1$$

- The recursion stops. We obtained all reflection coefficients, and also know that $H(z)$ was stable.

Discrete time filter structures

Matrix description of the allpass recursion



Verify that left and right produce the same relations between (a_i, b_i):

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} K & 1-K^2 \\ 1 & -K \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} b_2 \\ a_2 \end{bmatrix} = \frac{1}{1-K^2} \begin{bmatrix} 1 & -K \\ -K & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$$

Left is a “scattering matrix”, right a “chain matrix”.

The chain matrix plays a role in the reduction of a given allpass in cascade sections (the "analysis"). It was used implicitely in the lattice recursion, as follows.

Discrete time filter structures

Matrix interpretation of the lattice recursion

- Given $G_n(z) = \dfrac{a_{n,n} + a_{n,n-1}z^{-1} + \cdots + a_{n,1}z^{-n+1} + z^{-n}}{1 + a_{n,1}z^{-1} + \cdots + a_{n,n-1}z^{-n+1} + a_{n,n}z^{-n}} = \dfrac{\widetilde{A}_n(z)}{A_n(z)}$

  We now interprete the recursion (with $K_n = a_{n,n}$)

$$G_{n-1}(z) := z\,\frac{G_n(z) - K_n}{1 - K_n G_n(z)} = \frac{z\big(\widetilde{A}_n(z) - K_n A_n(z)\big)}{A_n(z) - K_n \widetilde{A}_n(z)}$$

- We know that $A_n(z)G_n(z) = \widetilde{A}_n(z)$: the response of the filter to an input $x[n] = [\boxed{1}, a_{n,1}, \cdots, a_{n,n-1}, a_{n,n}, 0 \cdots]$ is $y[n] = [\boxed{a_{n,n}}, a_{n,n-1}, \cdots, a_{n,1}, 1, 0 \cdots]$.

  We will create a system that maps $x[n]$ to $y[n]$. We start by analysis: reduce $x[n]$ and $y[n]$ by creating zeros. We will use the chain matrix.

$$\frac{1}{1-K_n^2}\begin{bmatrix} 1 & -K_n \\ -K_n & 1 \end{bmatrix}\begin{bmatrix} 1 & a_{n,1} & \cdots & a_{n,n-1} & a_{n,n} \\ a_{n,n} & a_{n,n-1} & \cdots & a_{n,1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{a_{n,1}-K_n a_{n,n-1}}{1-K_n^2} & \cdots & * & 0 \\ 0 & \frac{a_{n,n-1}-K_n a_{n,1}}{1-K_n^2} & \cdots & * & 1 \end{bmatrix}$$

The objective is to zero entry $a_{n,n}$, for this we need $K_n = a_{n,n}$. At the same time, the last entry in the first row will become zero as well.

Discrete time filter structures

Matrix interpretation of the lattice recursion

- Next, we can shift the second row to the left (an 'advance', i.e., multiplication with z, when viewed as an operator on sequences):

$$\begin{bmatrix} 1 & \\ & z \end{bmatrix} \begin{bmatrix} 1 & * & \cdots & * & 0 \\ 0 & * & \cdots & * & 1 \end{bmatrix} = \begin{bmatrix} 1 & * & \cdots & * & 0 \\ * & \cdots & * & 1 & 0 \end{bmatrix} \quad \leftrightarrow \quad \begin{bmatrix} A_{n-1}(z) \\ \widetilde{A}_{n-1}(z) \end{bmatrix}$$

(we can drop the last column with zeros)

- The result is two vectors corresponding to the polynomials

$$A_{n-1}(z) = \frac{A_n(z) - K_n \widetilde{A}_n(z)}{1 - K_n^2}, \qquad \widetilde{A}_{n-1}(z) = \frac{z\left(\widetilde{A}_n(z) - K_n A_n(z)\right)}{1 - K_n^2}$$

The ratio of the two polynomials (filter response) is exactly $G_{n-1}(z)$.

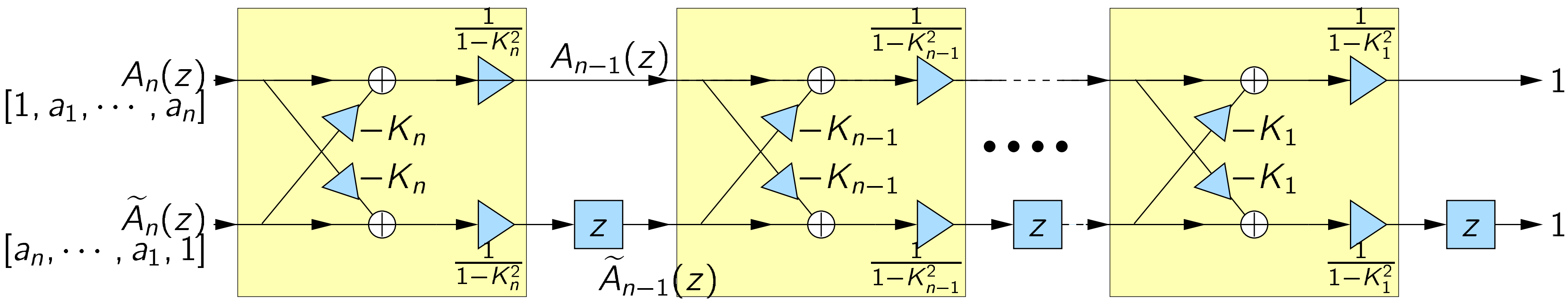
- The recursion continues by creating zeros in the second row, until

$$\begin{bmatrix} 1 & \\ & z \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

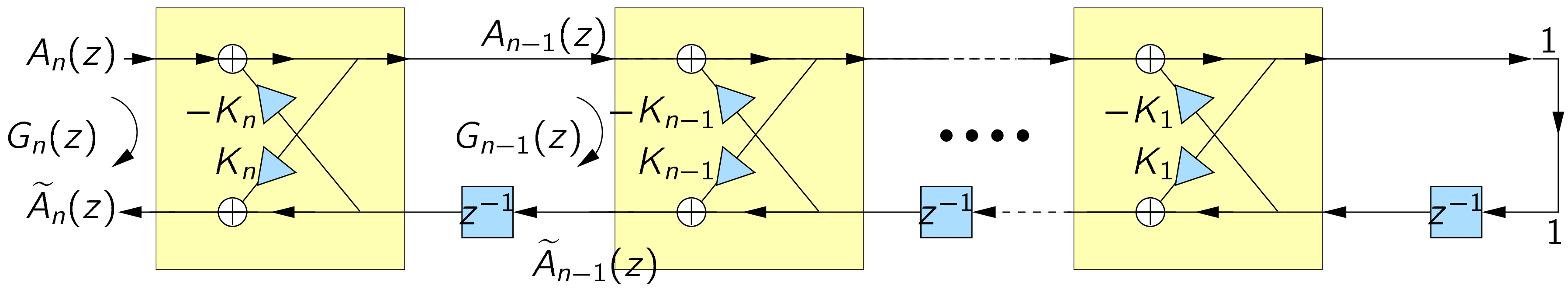
Discrete time filter structures

Illustration of the recursion:

Analysis



Synthesis



This is the requested filter: $G_n(z)$ maps $A_n(z)$ to $\widetilde{A}_n(z)$, i.e., $x[n]$ to $y[n]$.

Lattice filter

- We now want to realize an arbitrary (stable) filter $H(z) = \frac{B(z)}{A(z)}$. The approach is the same, but we also introduce a third series

$$B(z) \quad \leftrightarrow \quad z[n] = [\boxed{b_{n,1}}, b_{n,2}, \cdots, b_{n,n-1}, b_{n,n}]$$

which has to be made zero. We do not want to use additional delays.

- We use $\widetilde{A}_n(z)$, i.e., $y[n]$, to zero an entry of $z[n]$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -v_n & 1 \end{bmatrix} \begin{bmatrix} 1 & a_{n,1} & \cdots & a_{n,n-1} & a_{n,n} \\ a_{n,n} & a_{n,n-1} & \cdots & a_{n,1} & 1 \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n-1} & b_{n,n} \end{bmatrix} = \begin{bmatrix} 1 & a_{n,1} & \cdots & a_{n,n-1} & a_{n,n} \\ a_{n,n} & a_{n,n-1} & \cdots & a_{n,1} & 1 \\ * & * & \cdots & * & 0 \end{bmatrix}$$

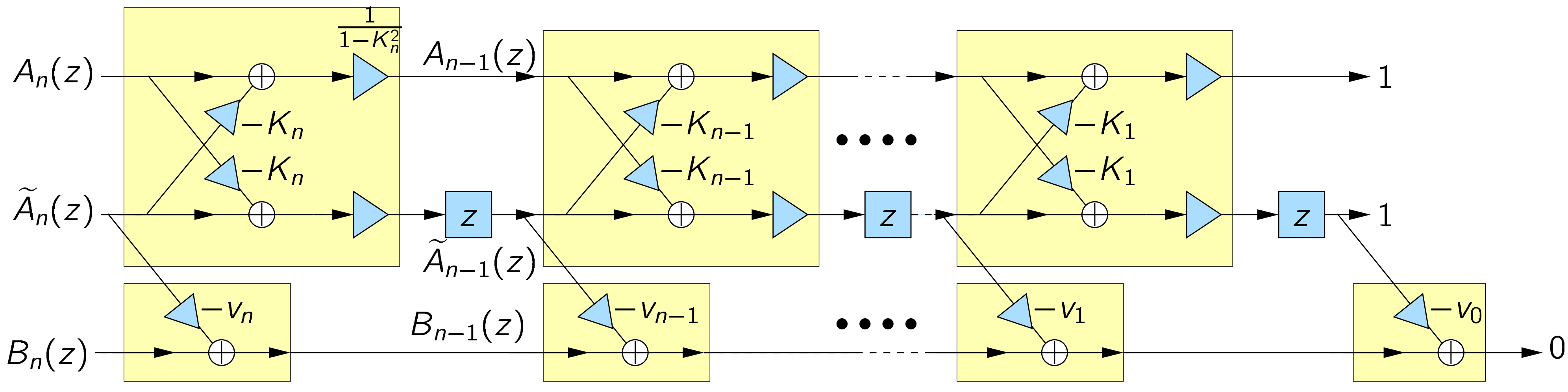
Hence, we choose $v_n = b_{n,n}$.

This corresponds to the zeroing of the highest coefficient of $B(z) - v_n\widetilde{A}(z)$

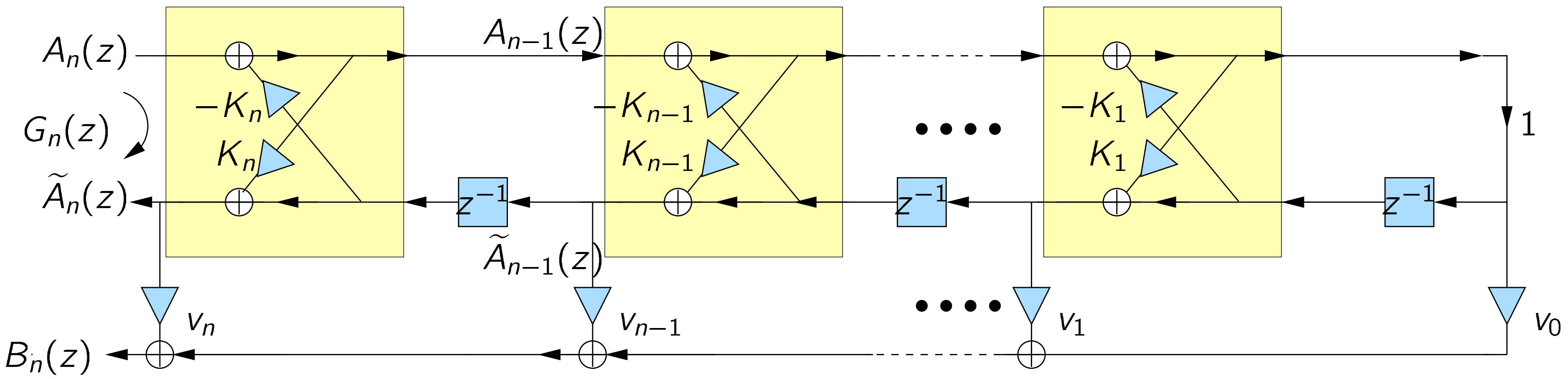
- Next, in the second row we zero $a_{n,n}$ against the first row, as before, and then shift the second row one notch to the left using an 'advance' z.

Discrete time filter structures

Analysis:



Synthesis:



Example

Derive a lattice filter realization for

$$H(z) = \frac{2.3 + 4.0z^{-1} + 3.6z^{-2} + 1.9z^{-3}}{1 + 0.9z^{-1} + 0.8z^{-2} + 0.3z^{-3}}$$

We have

$$\begin{aligned} A_3(z) &= 1 + 0.9z^{-1} + 0.8z^{-2} + 0.3z^{-3} \\ \widetilde{A}_3(z) &= 0.3 + 0.8z^{-1} + 0.9z^{-2} + z^{-3} \\ B_3(z) &= 2.3 + 4.0z^{-1} + 3.6z^{-2} + 1.9z^{-3} \end{aligned}$$

The recursion gives $v_3 = 1.9$, $K_3 = 0.3$, and

$$\begin{aligned} A_2(z) &= \frac{A_3(z) - K_3\widetilde{A}_3(z)}{1-K_3^2} &&= 1 + 0.72527z^{-1} + 0.5824z^{-2} \\ \widetilde{A}_2(z) &= \frac{z\left(\widetilde{A}_3(z) - K_3A_3(z)\right)}{1-K_3^2} &&= 0.5824 + 0.72527z^{-1} + z^{-2} \\ B_2(z) &= B_3(z) - v_3\widetilde{A}_3(z) &&= 1.73 + 2.48z^{-1} + 1.89z^{-2} \end{aligned}$$

Hence $v_2 = 1.89$, $K_2 = 0.5824$

Discrete time filter structures

Example (continued)

With $v_2 = 1.89$, $K_2 = 0.5824$ we obtain

$$\begin{aligned}
A_1(z) &= \tfrac{A_2(z) - K_2 \widetilde{A}_2(z)}{1 - K_2^2} &&= 1 + 0.4583 z^{-1} \\
\widetilde{A}_1(z) &= \tfrac{z\left(\widetilde{A}_2(z) - K_2 A_2(z)\right)}{1 - K_2^2} &&= 0.4583 + z^{-1} \\
B_1(z) &= B_2(z) - v_2 \widetilde{A}_2(z) &&= 0.6292 + 1.10924 z^{-1}
\end{aligned}$$

Next $v_1 = 1.10924$ and $K_1 = 0.4583$,

$$\begin{aligned}
A_0(z) &= \tfrac{A_1(z) - K_1 \widetilde{A}_1(z)}{1 - K_1^2} &&= 1 \\
\widetilde{A}_0(z) &= \tfrac{z\left(\widetilde{A}_1(z) - K_1 A_1(z)\right)}{1 - K_1^2} &&= 1 \\
B_0(z) &= B_1(z) - v_1 \widetilde{A}_1(z) &&= 0.12083
\end{aligned}$$

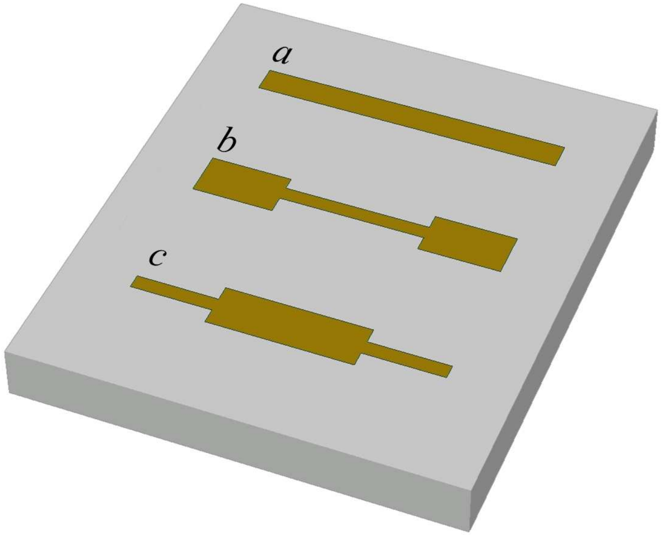
The last coefficient is $v_0 = 0.12083$.

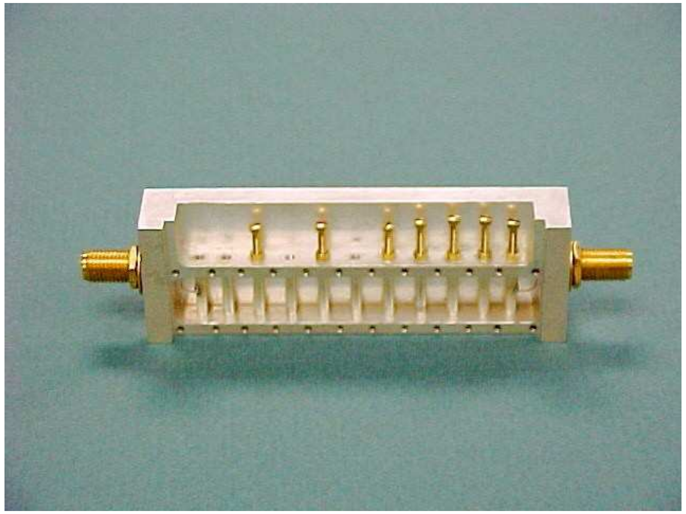
The filter is stable because during the recursion, all $|K_n| < 1$.

Microstrip filter

RF filters: microstrip and cavity filters

The allpass filter structure is also employed in RF microstrip filters. At the interface between wire segments of different width, reflections and transmissions occur (the width ratios determine the reflection coefficients). The overall structure is passive (ideally lossless). See ET4387 Passive components for microwave systems.





The structure is also used in models of earth layers (acoustic transmission/reflection in seismic/geophysic studies of the earth)