Signal Processing EE2S31

Digital Signal Processing
Lecture 9: Multirate systems

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Outline

- Overview and applications
- Sampling rate conversion
  - Decimation by factor M
  - Upsampling by a factor L
  - Sampling rate conversion by factor L/M
- Implementation of sampling rate conversion
  - Sampling rate conversion using polyphase filters
  - Multistage Sampling rate conversion

Overview and applications

Multirate systems

Multirate systems use different sampling rates at different stages of the digital processing.

Examples:

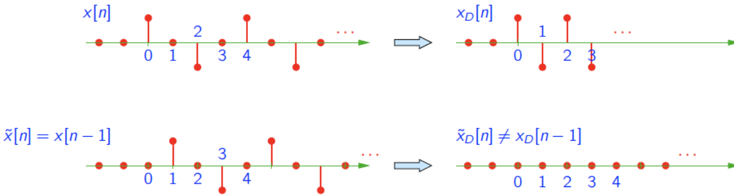
- telecommunication systems (speech, video)
- digital antialiasing (CD players)
- subband coding for speech and audio
- multimodal biomedical monitoring (EEG, ECG)

Decimation by a factor M

$$T_s' = MT \Rightarrow x_D[n] = x_a(nT_s') = x_a(nMT_s) = x[nM]$$

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow x_D[n] = x[Mn]$$

Decimation is a linear time-varying operation!



Decimantion by a factor M

$$p[\ell] = \frac{1}{M}\sum_{k=0}^{M-1} e^{j2\pi k\ell/M} = \begin{cases} 1, & \ell = 0, \pm M, \pm 2M, \cdots \\ 0, & \end{cases}$$

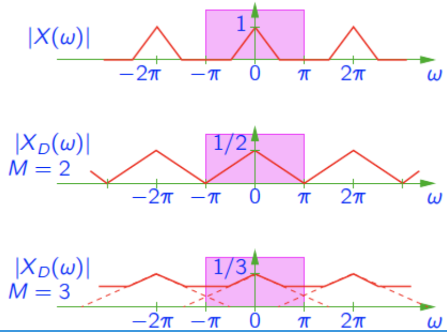
$$\begin{aligned}
X_D(z) &= \sum_{n=-\infty}^{\infty} x_D[n] z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x[nM] z^{-nM/M} \\
&= \sum_{\ell=-\infty}^{\infty} p[\ell]\, x[\ell]\, z^{-\ell/M} \\
&= \sum_{\ell=-\infty}^{\infty} \left\{ \frac{1}{M}\sum_{k=0}^{M-1} e^{j2\pi k\ell/M} \right\} x[\ell]\, z^{-\ell/M} \\
&= \frac{1}{M}\sum_{k=0}^{M-1} \sum_{\ell=-\infty}^{\infty} x[\ell] (z^{1/M} e^{-j2\pi k/M})^{-\ell} \\
&= \frac{1}{M}\sum_{k=0}^{M-1} X(z^{1/M} e^{-j2\pi k/M})
\end{aligned}$$

Decimation by a factor M

Spectrum of the decimated signal:

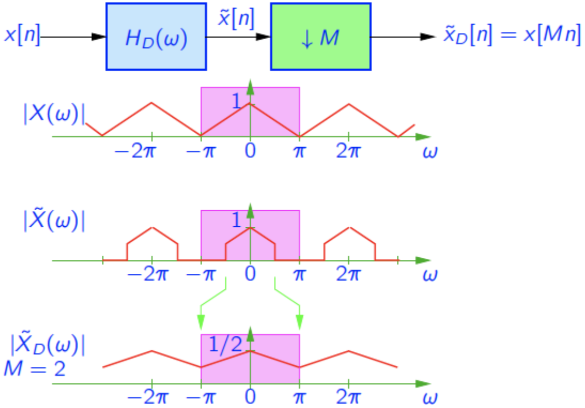
$$X_D(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{1/M}e^{-j2\pi k/M}), \qquad X_D(\omega) = \frac{1}{M}\sum_{k=0}^{M-1} X(\frac{\omega - 2\pi k}{M})$$

The spectrum of the decimated signal $X_D(\omega)$ consists of shifted and stretched copies of the original spectrum $X(\omega)$. (Note, $x[n]$ is digital, so $X(\omega)$ is already periodic!)



Decimantion by a factor M

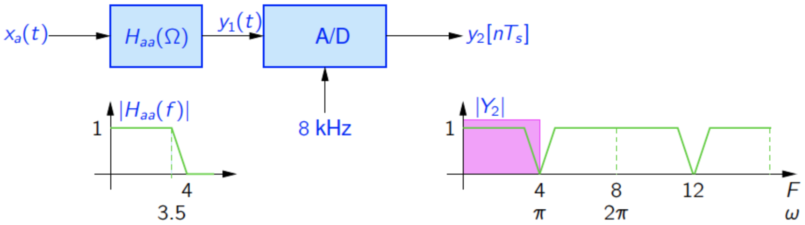
How to avoid aliasing?



We need to use an anti-aliasing decimationfilter (low-pass filter) with cut-off at $\frac{\pi}{M}$.

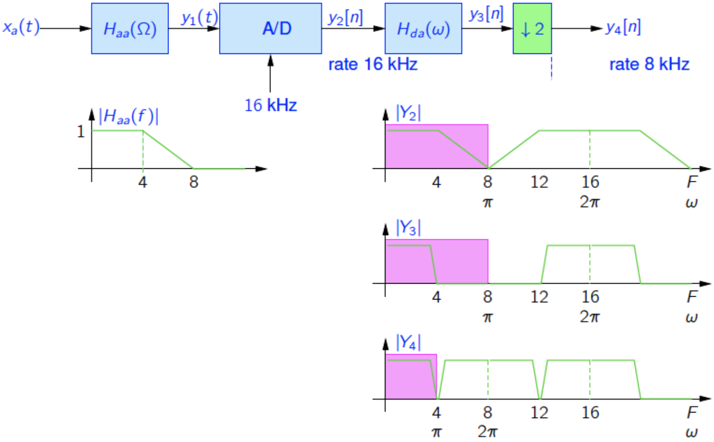
Example: audio signal

Let's assume that we have an audio signal where frequencies up to 3.5 kHz are important. We want to sample the signal at 8 kHz. In order to avoid aliasing, we need an anti-aliasing filter $H_{aa}(\Omega)$ with a transition band between 3.5 and 4 kHz. This is too sharp in analog domain!



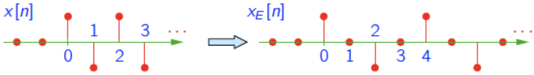
Example: audio signal

Alternative: let us sample with 16 kHz, filter in digital domain and then downsample! This allows an analog filter with reasonable transition band.



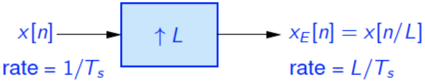
Interpolation with a factor L

Interpolation (=upsampling) with a factor L means to insert $L-1$ zeros:



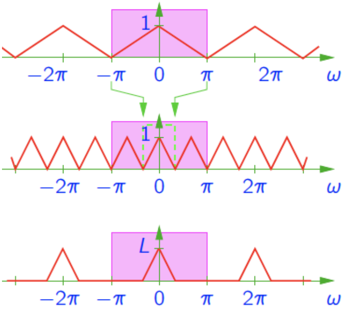
$$x_E[n] = \begin{cases} x[k], & n = kL \\ 0, & \text{otherwise} \end{cases}$$

The sampling rate of the signal increases by a factor L. There is no information loss!



Interpolation with a factor L

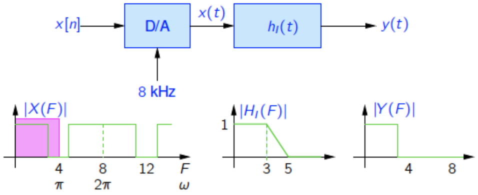
$$X_E(z) = X(z^L),\ X_E(\omega) = X(\omega L)$$



- The spectrum of the signal contracts with a factor L, and $L-1$ copies occur in the fundamental interval.
- We can remove the copies using a digital low-pass filter
- the overall system (i.e. sampling with $1/T_s$, upsampling and filtering) is then equivalent with a system sampled at a rate of L/T_s.

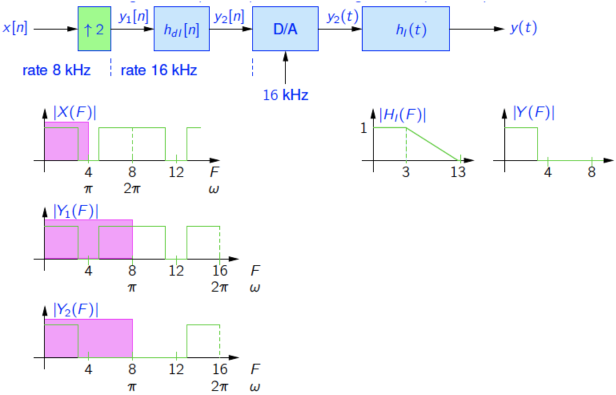
Example: audio signal

Let's assume that the audio signal now has to be converted back to analog and the highest frequency is 3 kHz, sample rate 8 kHz. After D/A convesion we need an image rejection (interpolation) filter with transition band between 3 – 5 kHz.



Example: audio signal

Alternative: Let us first upsample with a factor of 2, and then do D/A conversion!



Now the analog lowpass filter has a more reasonable transition band.

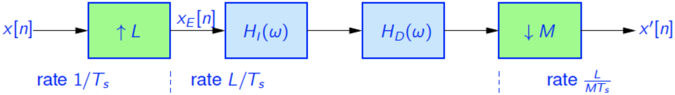
Exercise (exam question 2006)

A CD stores audio samples with a sampling rate of 44kHz. The highest audible frequency by the human ear is around 20 kHz.

- On the CD player, the text "2 times oversampled" is written. What does it mean exactly?
- Draw the block scheme of the reconstruction, including the frequency diagram of all signals.

Sampling rate conversion by factor L/M

We can combine upsampling with L and downsampling with M to implement a sampling rate conversion with any rational factor L/M.

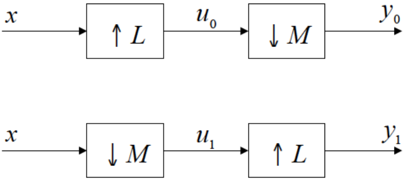


The two low-pass filters can be combined in one with a cut-off of $\omega_c = min(\pi/L, \pi/M)$.

Can we do the other way around, first decimate then interpolate? (Would that give the same output signal?)

Sampling rate conversion by factor L/M

Exercise 11.9:



General rule: $y_0 = y_1$ if and only if L and M are relative primes!

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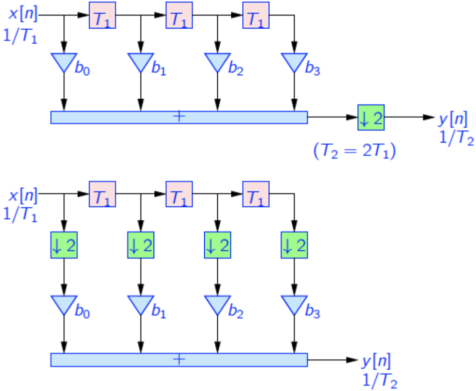
- **Implementation of sampling rate conversion**
  - Sampling rate conversion using polyphase filters
  - Multistage Sampling rate conversion

Implementation of sampling rate conversion

Goal

Efficiently implement multirate conversion systems

Example:



Note: In general, we cannot swap a filter and a sampling rate converter!

Implementation of sampling rate conversion

Objective

Let's modify the filter realisation such that filters and sampling rate converters can be swapped.

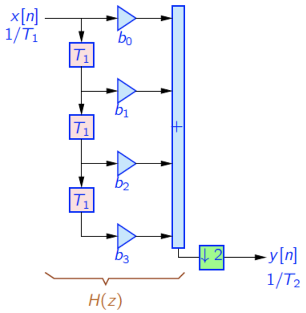
Outcome

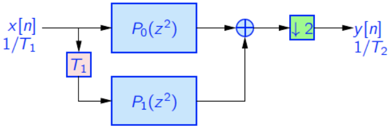
The resulting circuit is overall more efficient

Two ingredients:

- Polyphase filter structures
- Noble identities

Polyphase filter structures





$P_0(z) = b_0 + b_2 z^{-1}$
$P_1(z) = b_1 + b_3 z^{-1}$

$$
\begin{aligned}
H(z) &= b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} = \\
&= (b_0 + b_2 z^{-2}) + z^{-1}(b_1 + b_3 z^{-2}) = \\
&= P_0(z^2) + z^{-1} P_1(z^2)
\end{aligned}
$$

Polyphase filter structures

In general:

$$
\begin{aligned}
H(z) = b_0 \qquad & + b_M z^{-M} + \ldots \\
+ b_1 z^{-1} \qquad & + b_{M+1} z^{-M+1} + \ldots \\
+ b_{M-1} z^{M-1} & + b_{2M-1} z^{(-2M-1)} = \ldots
\end{aligned}
$$

$$
\begin{aligned}
H(z) = \qquad & [b_0 + b_M z^{-M} + \ldots] + \\
+ z^{-1} \quad & [b_1 + b_{M+1} z^{-M} + \ldots] + \\
+ z^{-M-1} & [b_{M-1} z^1 + b_{2M-1} z^{(-M)} + \ldots]
\end{aligned}
$$

This can be expressed using the *M-component polyphase representation*, with *polyphase components* $P_i(z)$:

$$H(z) = \sum_{i=0}^{M-1} z^{-i} P_i(z^M), \text{ where}$$

$$P_i(z^M) = \sum_{n=-\infty}^{\infty} b_{(nM+i)} z^{-n}$$

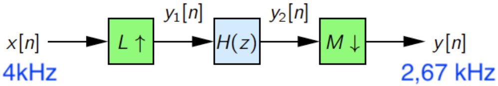
Noble identities

Identity 1:

Identity 2:

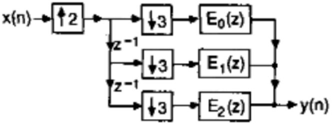
Example

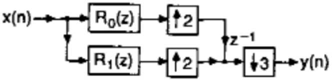
How to efficiently implement the following system:



Example

Step 1 (2 possibilities):



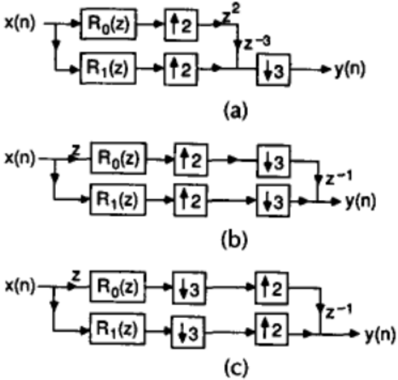


$(R_k(z) = E_{M-1-k}(z))$

Which one is more efficient (in terms of the rate at which the filters operate?)

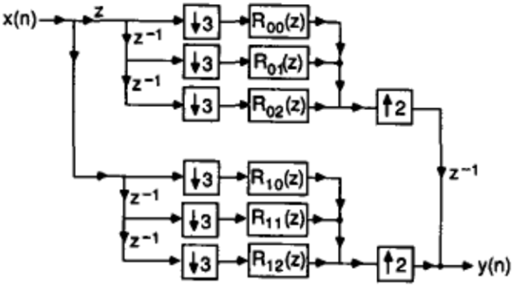
Example

Step 2 (let us work further on option 2):



Example

Most efficient implementation:



Now filters run at 4/3 kHz.

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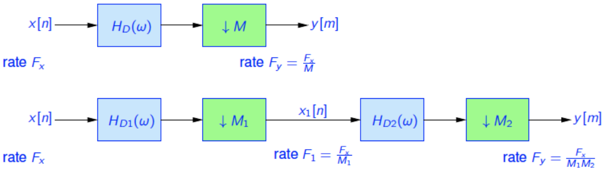
Multistage conversion

If M or L are too large, it is more efficient to implement conversion in multiple stages. This leads to lower order filters.



For example: $M = M_1 M_2$

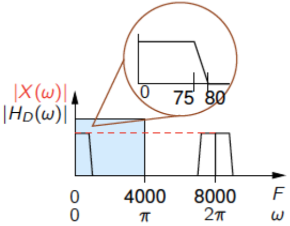
Multistage conversion

Example

Given an audio signal sampled at 8kHz. We want to keep the frequencies $0 - 80Hz$ and resample to 160Hz. Decimation factor is $M = 50$.

Let us assume the following filter specifications:

- passband: 0-75Hz, ripple $\delta_1 = 10^{-2}$
- transition band: 75-80Hz
- stopband: 80Hz-4000Hz, ripple $\delta_2 = 10^{-4}$



Heuristic formula to estimate filter order (Kaiser):

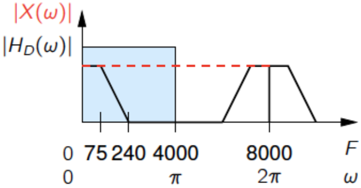
$$\hat{N} = \frac{-10\log(\delta_1\delta_2) - 13}{14,6\delta f}, \text{ where } \delta f = \frac{F_{stop} - F_{pass}}{F_x}$$

Example (continued)

Let us implement the filtering and downsampling in two stages, $M_1 = 25$ and $M_2 = 2$!

Stage 1:

- passband: 0-75Hz
- transition band: 75-240Hz
- ripples $\delta_1 = 0.5 \cdot 10^{-2}$ and $\delta_2 = 10^{-4}$

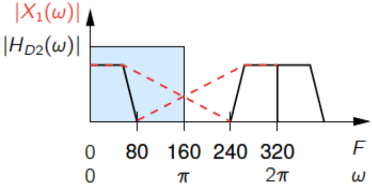


Here, $\delta f = \frac{165}{8000} = \frac{1}{48}$, therefore $\hat{N}_1 = 167$, which is much smaller than before.

Example (continued)

Stage 2:

- passband: 0-75Hz, ripple $\delta_1 = 10^{-2}$
- transition band: 75-80Hz
- ripples $\delta_1 = 0.5 \cdot 10^{-2}$ and $\delta_2 = 10^{-4}$



Here, $\hat{N}_2 = 220$, therefore, the a total $167 + 220 = 387$, which is much smaller than the 5151 using the single stage implementation.

Number of flops: $5151 \cdot 160 = 824$kflops vs
$167 \cdot 320 + 220 \cdot 160 = 88$kflops.