Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Signal Processing Systems

Partial exam EE2S31 SIGNAL PROCESSING
Part 1: 23 May 2023 (13:30-15:30)

Closed book; two sides of one A4 with handwritten notes permitted. No other tools except a basic pocket calculator permitted. Note the attached tables!

This exam consists of four questions (35 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.
Hint: Avoid losing too much time on detailed calculations, write down the general approach first.

Question 1 (11 points)

Given are two independent exponentially distributed random variables X and N, with probability density functions (pdfs)

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases} \qquad f_N(n) = \begin{cases} \lambda e^{-\lambda n} & \text{for } n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

where $\lambda > 0$. Although we are interested in X, we can only observe the process $Y = X + N$.

(a) Determine $\mathrm{E}[X^3]$.

(b) Derive that Y has a second-order Erlang distribution, i.e.

$$f_Y(y) = \begin{cases} \lambda^2 \, y \, e^{-\lambda y} & y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(c) Derive that the conditional pdf is

$$f_{Y|X}(y|x) = \begin{cases} \lambda e^{-\lambda(y-x)} & y \geq x \\ 0 & \text{otherwise.} \end{cases}$$

(d) Determine the joint pdf $f_{X,Y}(x,y)$.

(e) Determine $\hat{X}_{\mathrm{ML}}(Y)$, the ML estimate of X given Y.

(f) Determine $\hat{X}_{\mathrm{MMSE}}(Y)$, the MMSE estimate of X given Y.

(g) Determine the covariance $\mathrm{cov}[X,Y]$ and subsequently $\hat{X}_L(Y)$, the LMMSE estimate of X given Y.

Solution

(a) 2 pnt Use the MGF and its derivatives:

$$\phi_X(s) = \frac{\lambda}{\lambda - s}, \qquad \phi'_X(s) = \frac{\lambda}{(\lambda - s)^2}, \qquad \phi''_X(s) = \frac{2\lambda}{(\lambda - s)^3}, \qquad \phi'''_X(s) = \frac{6\lambda}{(\lambda - s)^4}$$

Then $\mathrm{E}[X^3] = \phi'''_X(s)\big|_{s=0} = \frac{6}{\lambda^3}$.

(b) 1 pnt The MGF of Y is the product of the MGF of X and that of N, resulting in

$$\phi_Y(s) = \left(\frac{\lambda}{\lambda - s}\right)^2$$

As seen in the table, this is the MGF of an Erlang distributed RV with $n = 2$.

(c) 1 pnt Given $X = x$, we have $Y = x + N$, so that $Y - x$ is distributed as N. Therefore,

$$f_{Y|X}(y|x) = f_N(y - x) = \begin{cases} \lambda e^{-\lambda(y-x)}, & y \geq x \\ 0, & \text{otherwise} \end{cases}$$

(d) 1.5 pnt

$$f_{X,Y}(x, y) = f_{Y|X}(y|x) f_X(x) = \lambda e^{-\lambda(y-x)} \lambda e^{-\lambda x} = \begin{cases} \lambda^2 e^{-\lambda y}, & y \geq 0, x \geq 0, x \leq y \\ 0, & \text{otherwise} \end{cases}$$

(e) 1.5 pnt The ML estimate maximizes the likelihood function over X,

$$\hat{X}(y) = \arg\max_x f_{Y|X}(y|x) = \arg\max_{0 \leq x \leq y} \lambda e^{-\lambda(y-x)}$$

Since $x \leq y$, this is maximal for $x = y$. Hence $\hat{X}(Y) = Y$.

(f) 2.5 pnt First derive the 'posterior' conditional probability

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \frac{\lambda^2 e^{-\lambda y}}{\lambda^2 \, y \, e^{-\lambda y}} = \begin{cases} \frac{1}{y}, & 0 \leq x \leq y \\ 0, & \text{otherwise} \end{cases}$$

(As a function of X, this is recognized as a uniform distribution.) Then the MMSE is

$$\hat{X}(y) = \mathrm{E}[X|Y = y] = \int x f_{X|Y}(x|y) \mathrm{d}x = \int_0^y x \frac{1}{y} \mathrm{d}x = \frac{1}{y} \left[\frac{1}{2} x^2\right]_0^y = \frac{1}{2} y$$

Therefore, $\hat{X}(Y) = \frac{1}{2} Y$.

(g) 1.5 pnt Using independence of X and N and properties of the exponential distribution

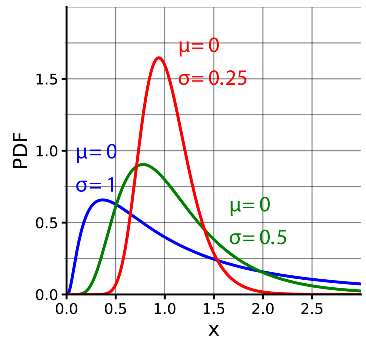
$$\mathrm{cov}[X, Y] = \mathrm{cov}[X, X + N] = \mathrm{cov}(X, X) + 0 = \mathrm{var}[X^2] = \frac{1}{\lambda^2}$$

$$\mu_X = \frac{1}{\lambda}, \qquad \mu_Y = \frac{2}{\lambda}, \qquad \mathrm{var}[Y] = \frac{2}{\lambda^2}$$

$$\begin{aligned} \hat{X}(Y) &= \frac{\mathrm{cov}[X, Y]}{\mathrm{var}[Y]} (Y - \mu_Y) + \mu_X \\ &= \frac{1}{2} (Y - \frac{2}{\lambda}) + \frac{1}{\lambda} \\ &= \frac{1}{2} Y . \end{aligned}$$

Not coincidentally equal to the MMSE estimator, since that one happened to be linear.

Question 2 (6 points)



In wireless communication, the received power fluctuates due to random obstruction by large buildings and hills. This "shadowing", expressed in dB, is often modeled as a normal (i.e. Gaussian) distribution. Equivalently, the received power has a log-normal distribution.

Let $X > 0$ be lognormal distributed with parameters (μ, σ^2), i.e., $\ln(X) = Y \sim \text{Gaussian}(\mu, \sigma^2)$.

(a) Derive that $\mathrm{E}[X^t] = e^{\mu t + \sigma^2 t^2/2}$, and determine expressions for $\mathrm{E}[X]$ and $\mathrm{var}[X]$.

(b) Let X_1 and X_2 be two independent log-normal random variables with parameters (μ_1, σ_1^2) and (μ_2, σ_2^2) respectively.

Show that their product $W = X_1 X_2$ is a log-normal random variable, and determine its parameters.

Let X be lognormal with parameters $\mu = 0$ and $\sigma = 0.25$.

(c) Determine $\mathrm{P}[X > 1.5]$.

(d) Use the Chebychev inequality to find an approximation for $\mathrm{P}[X > 1.5]$.

Solution

(a) 1.5 pnt With $X = e^Y$ and using the MGF of a Gaussian variable, the result follows directly:

$$\mathrm{E}[X^t] = \mathrm{E}[e^{tY}] = e^{\mu t + 1/2\sigma^2 t^2}$$

$$\begin{aligned}
\mathrm{E}[X] &= e^{\mu + \sigma^2/2} \\
\mathrm{E}[X^2] &= e^{2\mu + 2\sigma^2} \\
\mathrm{var}[X] &= \mathrm{E}[X^2] - \mathrm{E}[X]^2 = e^{2\mu+2\sigma^2} - (e^{\mu+\sigma^2/2})^2 = (e^{\sigma^2} - 1)(e^{2\mu+\sigma^2})
\end{aligned}$$

(b) 1.5 pnt

$$\ln(W) = \ln(X_1) + \ln(X_2) = Y_1 + Y_2$$

Since Y_1 and Y_2 are normal, and the sum of two normal random variables is again normal, then $Y = Y_1 + Y_2$ is also normal. It follows that W is lognormal.

Y has mean $\mu_Y = \mu_1 + \mu_2$ and variance $\sigma_Y^2 = \sigma_1^2 + \sigma_2^2$. Thus, the parameters of W are $(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

(c) 1.5 pnt Using the provided Table 4.1 on $\Phi(z)$,

$$\mathrm{P}[X > 1.5] = \mathrm{P}[\ln(X) > \ln(1.5)] = 1 - \Phi(\ln(1.5)/0.25) = 1 - \Phi(1.6219) = 1 - 0.9474 = 0.0526$$

(d) 1.5 pnt Using the result of (a),

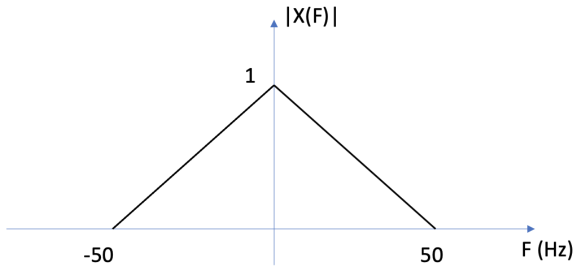
$$\begin{aligned}
\mu_X &= e^{0.25^2/2} = 1.0317 \\
\mathrm{var}[X] &= (e^{\sigma^2} - 1)(e^{2\mu+\sigma^2}) = 0.0687 \\
\mathrm{P}[X > 1.5] &= \mathrm{P}[X - \mu_X > 1.5 - \mu_X] \\
&< \mathrm{P}[|X - \mu_X| > 1.5 - \mu_X] \\
&< \frac{\mathrm{var}[X]}{(1.5 - \mu_X)^2} = 0.3133
\end{aligned}$$

Alternatively, first apply the log, and then use $\mu_Y = 0, \mathrm{var}[Y] = 0.25$:

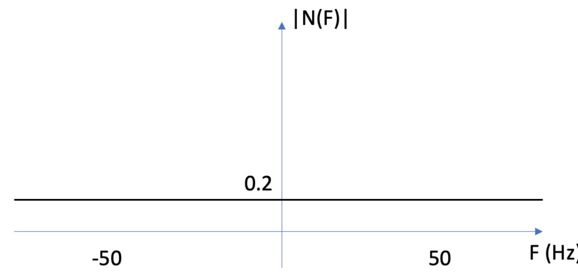
$$\begin{aligned} \mathrm{P}[X > 1.5] &= \mathrm{P}[\ln(X) > \ln(1.5)] \\ &= \mathrm{P}[Y > \ln(1.5)] \\ &< \frac{(0.25)^2}{\ln(1.5)^2} = 0.38 \end{aligned}$$

Question 3 (10 points)

We want to sample an analog signal $x(t)$, with frequency spectrum shown in figure a, below. The signal is contaminated with wide-band noise $n(t)$, shown in figure b.



a.



b.

(a) Design (i.e. characterize the frequency response of) an ideal anti-aliasing filter that removes the noise outside the bandwidth of the desired signal $x(t)$!

(b) What is the minimum sampling rate to avoid destructive aliasing?

(c) Assuming this sample rate, sketch the magnitude spectrum of the resulting sampled signal $x[n]$.

(d) What would be the minimum sampling rate for a signal $v(t)$, where

$$v(t) = x(t)\cos(400\pi t)$$

(e) Now let us reconstruct $\hat{x}_a(t)$ from its samples $x[n]$ (forget about $v(t)$) using sample-and-hold interpolation. Sketch the magnitude spectrum of $\hat{x}_a(t)$!

Hint: Recall that sample-and-hold interpolation is characterized in the frequency domain as $G_{SH}(F) = T\frac{\sin(\pi FT)}{\pi FT}e^{-2\pi F(T/2)}$.

(f) What is the highest frequency contained in $\hat{x}_a(t)$?

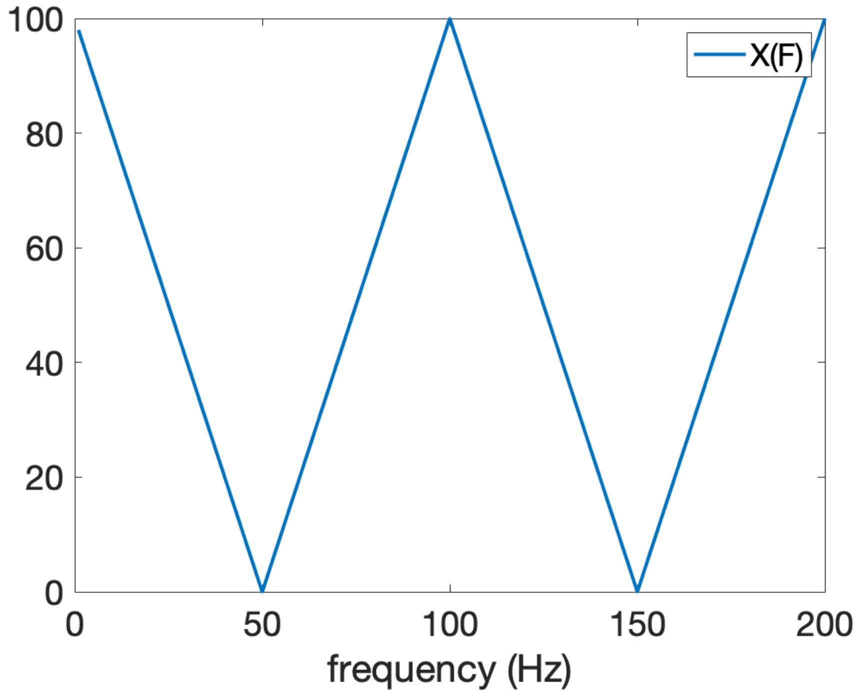
(g) At which frequencies will the spectrum of $\hat{x}_a(t)$ be equal to 0?

Solution

(a) 1 pnt

$$H(F) = \begin{cases} 1 & |F| \le 50Hz \\ 0 & \text{otherwise} \end{cases}$$

(b) 1 pnt 100 Hz.



(c) 1 pnt

(d) 3 pnt

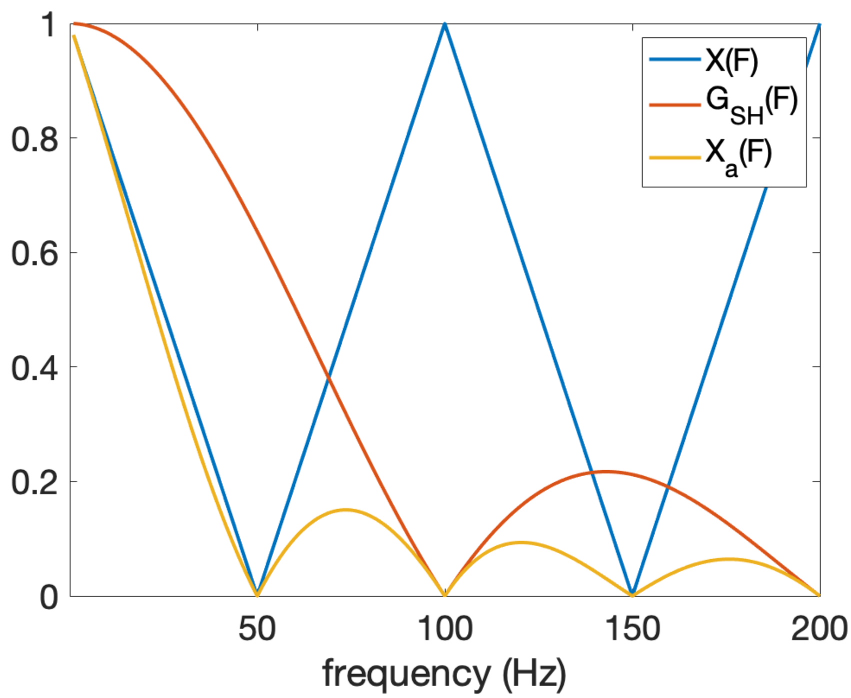
$$v(t) = x(t)\cos(400\pi t) = x(t)\cos(2\pi 200t), \text{ therefore, using the modulation property}$$
$$V(\Omega) = \frac{1}{2}(X(\Omega - 2\pi 200) + X(\Omega + 2\pi \cdot 200)), \text{ hence}$$
$$V(F) = \frac{1}{2}(X(F - 200) + X(F + 200))$$

Therefore, this is a bandpass signal with $F_H = 250$ Hz and $B = 100$Hz. The minimum sampling rate is:

$$F_s = \frac{2F_H}{k_{max}}, \text{ where}$$
$$k_{max} = \lfloor \frac{F_H}{B} \rfloor$$

Hence, $k_{max} = 2$ and $F_s = 250$ Hz.

(e) 2 pnt See below. Note that the $|X(F)|$ and $|G_{SH}(F)|$ are scaled with $1/F_s$ and F_s, respectively, for visualization purposes.

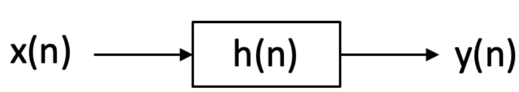


(f) 1 pnt As shown in the sketch, the high frequencies (i.e. shifted copies of the original spectrum) are significantly attenuated, although not completely removed. Hence, the reconstructed signal has infinitely high frequencies, although the magnitude of these frequencies tend to 0.

(g) 1 pnt The signal has zero magnitude at frequencies where the SH interpolator has zero-crossings, i.e. at integer multiples of $F_s = 100$ Hz, as well as where the sampled spectrum had 0 magnitude, i.e. at $50 + N \cdot 100$ Hz for any N integer.

Question 4 (8 points)

Given the digital system below, with input $x[n] = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$, filter impulse response $h[n] = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$, and output $y[n]$.



(a) Determine the output $y[n]$!

(b) The 4-point DFTs of $x[n]$ and $h[n]$ are equal to:

$$X[k] = \begin{bmatrix} 10 & -2+2j & -2 & -2-2j \end{bmatrix} \text{ and}$$
$$H[k] = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}$$

Determine the IDFT of $H[k] \cdot X[k]$!

(c) Compare the solution of part (a) and part (b). Are the samples of the two sequences the same? Explain why or why not!

(d) The input $x[n]$ can be considered as viewing an infinite sawtooth sequence through a rectangular window. Let's consider the use of the 4-point DFT given in part (b) to analyse the spectrum of the infinite sawtooth. The problem is that the finite windowing (with a window of length $N = 4$ in our case) effects the quality of the spectral estimation. This effect is best explained in frequency domain: the window main lobe limits the spectral resolution, while the sidelobes cause spectral leakage.

What can I do differently to reduce spectral leakage? Give two options!

Solution

(a) 2 pnt The output is the linear convolution of the input and the impulse response. This can be calculated as the product of the convolution matrix and the input sequence:

$$y[n] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 0 \end{bmatrix}$$

(b) 2 pnt The IDFT is computed as (matrix notation): $\frac{1}{N} W_N^H \mathbf{X}_N$. Here

$$\mathbf{X}_N = H(k) \cdot X(k) = \begin{bmatrix} 10 & 2-2j & -2 & 2+2j \end{bmatrix}$$

Then,

$$\frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 10 \\ 2-2j \\ -2 \\ 2+2j \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 2 \end{bmatrix}$$

(c) 2 pnt They are not the same. The linear convolution results in a sequence of $4 + 4 - 1 = 7$ samples. This needs at least a 7-point DFT to be represented without temporal aliasing. Instead, we multiplied the 4-point DFTs of the sequences. This causes temporal aliasing: the first sample becomes $0+3$, the second sample $0+4$ and the third sample $1+0$. In other words, the multiplication of DFT is equivalent to the circular convoltion of the sequences.

(d) 2 pnt Either take another window with smaller sidelobes (e.g. Hann) or take a larger window, i.e. more samples.