Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Signal Processing Systems

Partial exam EE2S31 SIGNAL PROCESSING
Part 1: 23 May 2023 (13:30-15:30)

Closed book; two sides of one A4 with handwritten notes permitted. No other tools except a basic pocket calculator permitted. Note the attached tables!

This exam consists of four questions (35 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.
Hint: Avoid losing too much time on detailed calculations, write down the general approach first.

Question 1 (11 points)

Given are two independent exponentially distributed random variables X and N, with probability density functions (pdfs)

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases} \qquad f_N(n) = \begin{cases} \lambda e^{-\lambda n} & \text{for } n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

where $\lambda > 0$. Although we are interested in X, we can only observe the process $Y = X + N$.

(a) Determine $\mathrm{E}[X^3]$.

(b) Derive that Y has a second-order Erlang distribution, i.e.

$$f_Y(y) = \begin{cases} \lambda^2\, y\, e^{-\lambda y} & y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(c) Derive that the conditional pdf is

$$f_{Y|X}(y|x) = \begin{cases} \lambda e^{-\lambda(y-x)} & y \geq x \\ 0 & \text{otherwise.} \end{cases}$$

(d) Determine the joint pdf $f_{X,Y}(x, y)$.

(e) Determine $\hat{X}_{\mathrm{ML}}(Y)$, the ML estimate of X given Y.

(f) Determine $\hat{X}_{\mathrm{MMSE}}(Y)$, the MMSE estimate of X given Y.

(g) Determine the covariance $\mathrm{cov}[X, Y]$ and subsequently $\hat{X}_L(Y)$, the LMMSE estimate of X given Y.

From Appendix A

Exponential (λ)

For $\lambda > 0$,

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = \frac{\lambda}{\lambda - s}$$

$$\mathrm{E}[X] = 1/\lambda$$

$$\mathrm{Var}[X] = 1/\lambda^2$$

Erlang (n, λ)

For $\lambda > 0$, and a positive integer n,

$$f_X(x) = \begin{cases} \dfrac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = \left(\frac{\lambda}{\lambda - s}\right)^n$$

$$\mathrm{E}[X] = n/\lambda$$

$$\mathrm{Var}[X] = n/\lambda^2$$

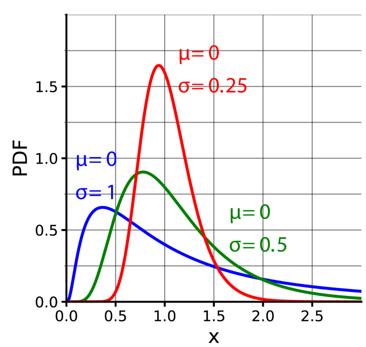
Gaussian (μ, σ)

For constants $\sigma > 0$, $-\infty < \mu < \infty$,

$$f_X(x) = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \qquad \phi_X(s) = e^{s\mu + s^2\sigma^2/2}$$

$$\mathrm{E}[X] = \mu$$

$$\mathrm{Var}[X] = \sigma^2$$



Question 2 (6 points)

In wireless communication, the received power fluctuates due to random obstruction by large buildings and hills. This "shadowing", expressed in dB, is often modeled as a normal (i.e. Gaussian) distribution. Equivalently, the received power has a log-normal distribution.

Let $X > 0$ be lognormal distributed with parameters (μ, σ^2), i.e., $\ln(X) = Y \sim \text{Gaussian}(\mu, \sigma^2)$.

(a) Derive that $\mathrm{E}[X^t] = e^{\mu t + \sigma^2 t^2/2}$, and determine expressions for $\mathrm{E}[X]$ and $\mathrm{var}[X]$.

(b) Let X_1 and X_2 be two independent log-normal random variables with parameters (μ_1, σ_1^2) and (μ_2, σ_2^2) respectively.

Show that their product $W = X_1 X_2$ is a log-normal random variable, and determine its parameters.

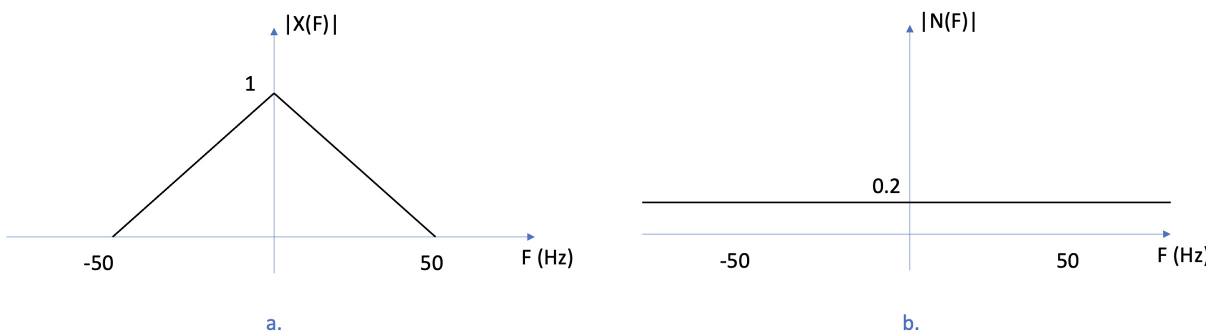
Let X be lognormal with parameters $\mu = 0$ and $\sigma = 0.25$.

(c) Determine $\mathrm{P}[X > 1.5]$.

(d) Use the Chebychev inequality to find an approximation for $\mathrm{P}[X > 1.5]$.

Question 3 (10 points)

We want to sample an analog signal $x(t)$, with frequency spectrum shown in figure a, below. The signal is contaminated with wide-band noise $n(t)$, shown in figure b.



(a) Design (i.e. characterize the frequency response of) an ideal anti-aliasing filter that removes the noise outside the bandwidth of the desired signal $x(t)$!

(b) What is the minimum sampling rate to avoid destructive aliasing?

(c) Assuming this sample rate, sketch the magnitude spectrum of the resulting sampled signal $x[n]$.

(d) What would be the minimum sampling rate for a signal $v(t)$, where

$$v(t) = x(t)\cos(400\pi t)$$

(e) Now let us reconstruct $\hat{x}_a(t)$ from its samples $x[n]$ (forget about $v(t)$) using sample-and-hold interpolation. Sketch the magnitude spectrum of $\hat{x}_a(t)$!

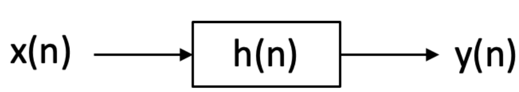
Hint: Recall that sample-and-hold interpolation is characterized in the frequency domain as $G_{SH}(F) = T\frac{\sin(\pi FT)}{\pi FT}e^{-2\pi F(T/2)}$.

(f) What is the highest frequency contained in $\hat{x}_a(t)$?

(g) At which frequencies will the spectrum of $\hat{x}_a(t)$ be equal to 0?

Question 4 (8 points)

Given the digital system below, with input $x[n] = \begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}$, filter impulse response $h[n] = \begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}$, and output $y[n]$.



(a) Determine the output $y[n]$!

(b) The 4-point DFTs of $x[n]$ and $h[n]$ are equal to:

$$X[k] = \begin{bmatrix} 10 & -2+2j & -2 & -2-2j \end{bmatrix} \text{ and}$$
$$H[k] = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}$$

Determine the IDFT of $H[k] \cdot X[k]$!

(c) Compare the solution of part (a) and part (b). Are the samples of the two sequences the same? Explain why or why not!

(d) The input $x[n]$ can be considered as viewing an infinite sawtooth sequence through a rectangular window. Let's consider the use of the 4-point DFT given in part (b) to analyse the spectrum of the infinite sawtooth. The problem is that the finite windowing (with a window of length $N = 4$ in our case) effects the quality of the spectral estimation. This effect is best explained in frequency domain: the window main lobe limits the spectral resolution, while the sidelobes cause spectral leakage.

What can I do differently to reduce spectral leakage? Give two options!

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.00	0.5000	0.50	0.6915	1.00	0.8413	1.50	0.9332	2.00	0.97725	2.50	0.99379
0.01	0.5040	0.51	0.6950	1.01	0.8438	1.51	0.9345	2.01	0.97778	2.51	0.99396
0.02	0.5080	0.52	0.6985	1.02	0.8461	1.52	0.9357	2.02	0.97831	2.52	0.99413
0.03	0.5120	0.53	0.7019	1.03	0.8485	1.53	0.9370	2.03	0.97882	2.53	0.99430
0.04	0.5160	0.54	0.7054	1.04	0.8508	1.54	0.9382	2.04	0.97932	2.54	0.99446
0.05	0.5199	0.55	0.7088	1.05	0.8531	1.55	0.9394	2.05	0.97982	2.55	0.99461
0.06	0.5239	0.56	0.7123	1.06	0.8554	1.56	0.9406	2.06	0.98030	2.56	0.99477
0.07	0.5279	0.57	0.7157	1.07	0.8577	1.57	0.9418	2.07	0.98077	2.57	0.99492
0.08	0.5319	0.58	0.7190	1.08	0.8599	1.58	0.9429	2.08	0.98124	2.58	0.99506
0.09	0.5359	0.59	0.7224	1.09	0.8621	1.59	0.9441	2.09	0.98169	2.59	0.99520
0.10	0.5398	0.60	0.7257	1.10	0.8643	1.60	0.9452	2.10	0.98214	2.60	0.99534
0.11	0.5438	0.61	0.7291	1.11	0.8665	1.61	0.9463	2.11	0.98257	2.61	0.99547
0.12	0.5478	0.62	0.7324	1.12	0.8686	1.62	0.9474	2.12	0.98300	2.62	0.99560
0.13	0.5517	0.63	0.7357	1.13	0.8708	1.63	0.9484	2.13	0.98341	2.63	0.99573
0.14	0.5557	0.64	0.7389	1.14	0.8729	1.64	0.9495	2.14	0.98382	2.64	0.99585
0.15	0.5596	0.65	0.7422	1.15	0.8749	1.65	0.9505	2.15	0.98422	2.65	0.99598
0.16	0.5636	0.66	0.7454	1.16	0.8770	1.66	0.9515	2.16	0.98461	2.66	0.99609
0.17	0.5675	0.67	0.7486	1.17	0.8790	1.67	0.9525	2.17	0.98500	2.67	0.99621
0.18	0.5714	0.68	0.7517	1.18	0.8810	1.68	0.9535	2.18	0.98537	2.68	0.99632
0.19	0.5753	0.69	0.7549	1.19	0.8830	1.69	0.9545	2.19	0.98574	2.69	0.99643
0.20	0.5793	0.70	0.7580	1.20	0.8849	1.70	0.9554	2.20	0.98610	2.70	0.99653
0.21	0.5832	0.71	0.7611	1.21	0.8869	1.71	0.9564	2.21	0.98645	2.71	0.99664
0.22	0.5871	0.72	0.7642	1.22	0.8888	1.72	0.9573	2.22	0.98679	2.72	0.99674
0.23	0.5910	0.73	0.7673	1.23	0.8907	1.73	0.9582	2.23	0.98713	2.73	0.99683
0.24	0.5948	0.74	0.7704	1.24	0.8925	1.74	0.9591	2.24	0.98745	2.74	0.99693
0.25	0.5987	0.75	0.7734	1.25	0.8944	1.75	0.9599	2.25	0.98778	2.75	0.99702
0.26	0.6026	0.76	0.7764	1.26	0.8962	1.76	0.9608	2.26	0.98809	2.76	0.99711
0.27	0.6064	0.77	0.7794	1.27	0.8980	1.77	0.9616	2.27	0.98840	2.77	0.99720
0.28	0.6103	0.78	0.7823	1.28	0.8997	1.78	0.9625	2.28	0.98870	2.78	0.99728
0.29	0.6141	0.79	0.7852	1.29	0.9015	1.79	0.9633	2.29	0.98899	2.79	0.99736
0.30	0.6179	0.80	0.7881	1.30	0.9032	1.80	0.9641	2.30	0.98928	2.80	0.99744
0.31	0.6217	0.81	0.7910	1.31	0.9049	1.81	0.9649	2.31	0.98956	2.81	0.99752
0.32	0.6255	0.82	0.7939	1.32	0.9066	1.82	0.9656	2.32	0.98983	2.82	0.99760
0.33	0.6293	0.83	0.7967	1.33	0.9082	1.83	0.9664	2.33	0.99010	2.83	0.99767
0.34	0.6331	0.84	0.7995	1.34	0.9099	1.84	0.9671	2.34	0.99036	2.84	0.99774
0.35	0.6368	0.85	0.8023	1.35	0.9115	1.85	0.9678	2.35	0.99061	2.85	0.99781
0.36	0.6406	0.86	0.8051	1.36	0.9131	1.86	0.9686	2.36	0.99086	2.86	0.99788
0.37	0.6443	0.87	0.8078	1.37	0.9147	1.87	0.9693	2.37	0.99111	2.87	0.99795
0.38	0.6480	0.88	0.8106	1.38	0.9162	1.88	0.9699	2.38	0.99134	2.88	0.99801
0.39	0.6517	0.89	0.8133	1.39	0.9177	1.89	0.9706	2.39	0.99158	2.89	0.99807
0.40	0.6554	0.90	0.8159	1.40	0.9192	1.90	0.9713	2.40	0.99180	2.90	0.99813
0.41	0.6591	0.91	0.8186	1.41	0.9207	1.91	0.9719	2.41	0.99202	2.91	0.99819
0.42	0.6628	0.92	0.8212	1.42	0.9222	1.92	0.9726	2.42	0.99224	2.92	0.99825
0.43	0.6664	0.93	0.8238	1.43	0.9236	1.93	0.9732	2.43	0.99245	2.93	0.99831
0.44	0.6700	0.94	0.8264	1.44	0.9251	1.94	0.9738	2.44	0.99266	2.94	0.99836
0.45	0.6736	0.95	0.8289	1.45	0.9265	1.95	0.9744	2.45	0.99286	2.95	0.99841
0.46	0.6772	0.96	0.8315	1.46	0.9279	1.96	0.9750	2.46	0.99305	2.96	0.99846
0.47	0.6808	0.97	0.8340	1.47	0.9292	1.97	0.9756	2.47	0.99324	2.97	0.99851
0.48	0.6844	0.98	0.8365	1.48	0.9306	1.98	0.9761	2.48	0.99343	2.98	0.99856
0.49	0.6879	0.99	0.8389	1.49	0.9319	1.99	0.9767	2.49	0.99361	2.99	0.99861

Table 4.1 The standard normal CDF $\Phi(z)$.

Table 5.1 Basic Properties of Fourier Transform

	Time Domain	Frequency Domain
Signals and constants	$x(t)$, $y(t)$, $z(t)$, α, β	$X(\Omega), Y(\Omega), Z(\Omega)$
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(\Omega) + \beta Y(\Omega)$
Expansion/contraction in time	$x(\alpha t)$, $\alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert} X\left(\frac{\Omega}{\alpha}\right)$
Reflection	$x(-t)$	$X(-\Omega)$
Parseval's energy relation	$E_x = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	$E_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(\Omega)\rvert^2 d\Omega$
Duality	$X(t)$	$2\pi x(-\Omega)$
Time differentiation	$\frac{d^n x(t)}{dt^n}, n \geq 1$, integer	$(j\Omega)^n X(\Omega)$
Frequency differentiation	$-jtx(t)$	$\frac{dX(\Omega)}{d\Omega}$
Integration	$\int_{-\infty}^{t} x(t')dt'$	$\frac{X(\Omega)}{j\Omega} + \pi X(0)\delta(\Omega)$
Time shifting	$x(t - \alpha)$	$e^{-j\alpha\Omega} X(\Omega)$
Frequency shifting	$e^{j\Omega_0 t} x(t)$	$X(\Omega - \Omega_0)$
Modulation	$x(t)\cos(\Omega_c t)$	$0.5[X(\Omega - \Omega_c) + X(\Omega + \Omega_c)]$
Periodic signals	$x(t) = \sum_k X_k e^{jk\Omega_0 t}$	$X(\Omega) = \sum_k 2\pi X_k \delta(\Omega - k\Omega_0)$
Symmetry	$x(t)$ real	$\lvert X(\Omega)\rvert = \lvert X(-\Omega)\rvert$ $\angle X(\Omega) = -\angle X(-\Omega)$
Convolution in time	$z(t) = [x * y](t)$	$Z(\Omega) = X(\Omega)Y(\Omega)$
Windowing/Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}[X * Y](\Omega)$
Cosine transform	$x(t)$ even	$X(\Omega) = \int_{-\infty}^{\infty} x(t)\cos(\Omega t)dt$, real
Sine transform	$x(t)$ odd	$X(\Omega) = -j\int_{-\infty}^{\infty} x(t)\sin(\Omega t)dt$, imaginary