Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Signal Processing Systems

Resit exam EE2S31 SIGNAL PROCESSING
20 July 2023 13:30–16:30

Closed book; two sides of one A4 with handwritten notes permitted. No other tools except a basic pocket calculator permitted. Note the attached tables!

This exam consists of five questions (36 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.
Hint: Avoid losing too much time on detailed calculations, write down the general approach first.

Question 1 (11 points)

We want to estimate an unknown audio channel (i.e. FIR filter) $h[n]$ by sending a known audio signal $x[n]$ through it, and measuring the output $y[n] = x[n] * h[n]$. Our goal is to locate the first peak in $h[n]$.

The length of $x[n]$ is $N_x = 500$. The sample rate of the receiver is 20 kHz. We take $N_y = 2000$ samples of the output signal $y[n]$. Let us assume that the channel impulse response is of length L.

We will estimate $h[n]$ in the frequency domain, using the expression $H(\omega) = \frac{Y(\omega)}{X(\omega)}$. Considering the sampling rate of the receiver, the distance between two consecutive samples of $h[n]$ is 1/20 kHz = 0.05 ms. Therefore, we say that we can locate the first peak with 0.05 ms resolution.

(a) Give the formula for the relationship between N_x, N_y and L and determine the value L.

(b) How can you use the DFT and the inverse DFT (IDFT) to estimate $h[n]$? Explain the steps in detail!

In an alternative time-domain approach, we propose a different estimator

$$\hat{h}_2[n] = \frac{1}{\alpha}\, x[-n] * y[n]$$

where α is a scaling.

(c) Motivate this approach. For this, express $\hat{h}_2[n]$ in terms of $h[n]$ and use terminology such as autocorrelation.

(d) Determine a suitable value for α (motivate).

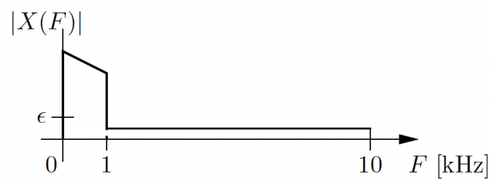
(e) Compare the computational complexity of both methods. For the frequency-domain approach, assume that we use a radix-2 FFT algorithm to compute the DFT and IDFT. Indicate the number of additions and multiplications needed for each approach.

Returning to the frequency-domain approach, note that there may be frequencies for which $|X(\omega)|$ is very small (let's say smaller than a threshold ϵ). For these frequencies, we simply take $\hat{H}(\omega) = 0$. The channel estimate is then

$$\hat{H}(\omega) = \frac{Y(\omega)}{X(\omega)} G(\omega), \qquad \text{where } G(\omega) = \begin{cases} 1 & \text{if } |X(\omega)| > \epsilon \\ 0 & \text{otherwise} \end{cases}$$

In the noiseless case, this means that $\hat{H}(\omega) = H(\omega)G(\omega)$.

(f) Express $\hat{h}[n]$ in terms of $h[n]$. Given the spectrum of $x[n]$ in the figure below, what is the effect of the thresholding (described above) on the channel estimate $\hat{h}[n]$?



(g) Determine $g[n]$, the inverse DTFT of $G(\omega)$ and make a sketch!

(h) Considering $g[n]$, what is the resolution now for locating the first peak?

Solution

1p (a) $N_y = N_x + L - 1 \quad \Rightarrow \quad L = N_y - N_x + 1 = 1501$.

2p (b) First zero padding of $x[n]$ such that it has the same length as $y[n]$, i.e. $N_y = 2000$. Next, take the DFT of both $x[n]$ and $y[n]$, resulting in $X[k]$ and $Y[k]$, each of length N_y. Then, $\hat{H}[k] = Y[k]/X[k]$, for all k. Finally, IDFT on $\hat{H}[k]$ to obtain $\hat{h}[n]$. This should be cut off at $L = 1501$ samples.

1.5p (c) Inserting $y[n] = x[n] * h[n]$, we obtain $\hat{h}_2[n] = \frac{1}{\alpha} x[n] * x[-n] * h[n] = \frac{1}{\alpha} \hat{R}_X[n] * h[n]$, where $\hat{R}_X[n] = x[n] * x[-n]$ is the "deterministic autocorrelation function" of the input. Ideally, it is close to a delta spike. In that case we would have $\hat{h}_2[n] = h[n]$.

0.5p (d) To normalize the result, we could set $\alpha = \hat{R}_X[0]$. This ensures that $\hat{h}[n] \approx h[n]$ in case $\hat{R}_X[n]$ is close to a delta spike. Alternatively, we could set $\alpha = \sum_n \hat{R}_X[n]$, which ensures that $\sum \hat{h}[n] \approx \sum h[n]$.

2p (e) The FFT has complexity $N_y \log_2(N_y) \approx 20,000$ operations, we need three of these. The pointwise division in frequency domain has a complexity of $N_y = 2000$ multiplications. In total, we need about $62,000$ operations.

In the time-domain approach, each sample of $\hat{h}_2[n]$ requires $N_x = 500$ multiplications and additions (ignoring edge effects!), and we need $L = 1501$ of these. In total, that is order $750,000$ multiplications and the same number of additions.

1p (f) $\hat{h} = h[n] * g[n]$. In frequency domain, $G(\omega)$ is an ideal lowpass filter with cut-off at 1 kHz, so the channel is lowpass filtered. In time domain, that results in a convolution of $h[n]$ with a sinc function.

2p (g) With $\omega_c = 0.1\pi$, the IDTFT of $G(\omega)$ is

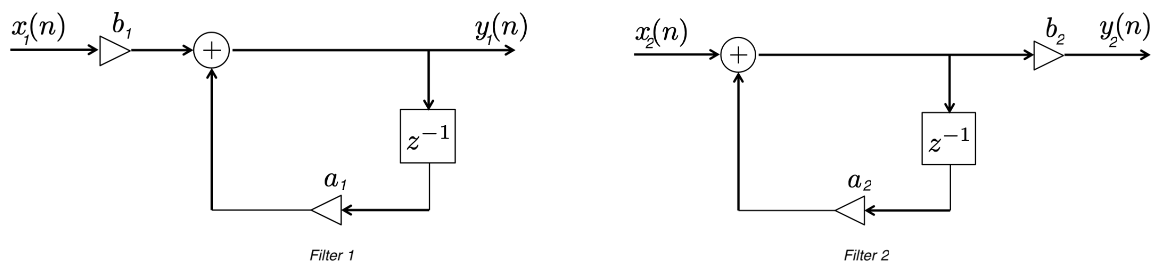
$$g[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{1}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{-\omega_c}^{\omega_c} = \frac{\sin(\omega_c n)}{\pi n}$$

(plot)

1p (h) The first zero crossing of $g[n]$ is at $n = 10$. The main lobe width of the sinc function is therefore estimated at 10 samples. This means that each sample of $h[n]$ is smeared out over approximately 10 samples. The effective resolution to locate the main peak in $\hat{h}[n]$ is therefore $10 \times 0.05 = 0.5$ ms.

Question 2 (8 points)

Consider the following first-order IIR filter realizations.



(a) Assuming that the multipliers $a_1 = a_2$ and the inputs $x_1(n) = x_2(n)$, for which value of the multiplier b_1 does Filter 1 have the same output as Filter 2? Express it in terms of the multiplier b_2!

(b) Give a formula for the transfer function of Filter 1!

(c) Give a formula for the impulse response of the filter in Filter 1!

In a practical scenario we have to quantize the outputs of the multipliers. Assume that the quantizers we use are uniform midtread quantizers with stepsize Δ. Assuming Δ is small enough, the quantization error can be modeled as an additive noise signal $z(n)$, which is a realization of an uncorrelated wide-sense stationary process, having a uniform distribution over the interval $[-\Delta/2, \Delta/2]$.

(d) Compute the total quantization noise power at the output of Filter 1!

(e) Assuming again that $a_1 = a_2$, for which value of b_2 does Filter 2 have the same quantization noise power at the output as Filter 1?

Solution

1p (a) $b_1 = b_2$

1p (b) $H_1(z) = \frac{b_1}{1-a_1 z^{-1}}$

1p (c) $h_1(n) = b_1 a_1^n u(n)$

3p (d) Both noise sources go through the feedback loop. The feedback loop has an impulse response $g(n) = a_1^n u(n)$. Hence, the total quantization noise power at the output is given by

$$\sigma_{q1}^2 = 2\,\frac{\Delta^2}{12}\,|g(n)|^2 = 2\,\frac{\Delta^2}{12}\,\frac{1}{1-a_1^2}$$

2p (e) The noise source due to the multiplier b_2 appears directly at the output of the filter. Therefore, the total quantization noise power at the output is given by

$$\sigma_{q2}^2 = \frac{\Delta^2}{12}(1 + |h_2(n)|^2) = \frac{\Delta^2}{12}(1 + \frac{b_2}{1 - a_2^2}).$$

This is equal to $2\,\frac{\Delta^2}{12}\,\frac{1}{1-a^2}$ if (using the notation $a_1 = a_2 = a$):

$$\begin{aligned} 2\frac{1}{1-a^2} &= 1 + \frac{b_2^2}{1-a^2} \\ \frac{2}{1-a^2} &= \frac{1-a^2+b_2^2}{1-a^2} \\ 2 &= 1 - a^2 + b_2^2 \\ b_2 &= \sqrt{1+a^2} \end{aligned}$$

Question 3 (6 points)

A joint probability density function of the random variables X and Y is given by

$$f_{X,Y}(X,Y) = \begin{cases} \dfrac{x^4}{2} & \text{for } 0 \le x \le 1,\ 0 \le y \le x^2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the marginal pdfs of the random variables X and Y.

(b) Compute the conditional pdf $f_{X|Y}(x|y)$.

For the remainder of this question, assume that $f_{X|Y}(x|y)$ is given by

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{x^2}{2(1-y^{3/2})} & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and that $f_X(x)$ is given by

$$f_X(x) = \begin{cases} \dfrac{5x^4}{2} & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

In an experiment, we observe a realization of random variable Y, while we want to make an estimate of X. To do so, we determine two different estimators for X.

(c) Calculate $\hat{X}_B$, the "blind" estimate.

(d) Calculate $\hat{X}_M(y)$, the MMSE estimate of X given a single observation y of Y.

(e) Explain in words which of the two estimators from Question (c) and (d) is better. Also define what you mean by "better".

(f) Under which conditions are the two estimators in Question (c) and (d) equal?

Solution

2p (a) [The given joint-pdf is not correct as it doesn't integrate to 1; sorry for the confusion!]

$$f_X(x) = \int_{y=0}^{x^2} \frac{x^4}{2} \mathrm{d}y = \left[\frac{x^4 y}{2}\right]_{y=0}^{y=x^2} = \frac{x^6}{2}$$

Therefore,

$$f_X(x) = \begin{cases} \frac{x^6}{2} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{x=\sqrt{y}}^{1} \frac{x^4}{2} \mathrm{d}x = \left[\frac{x^5}{10}\right]_{x=\sqrt{y}}^{x=1} = \frac{1}{10}(1-y^{5/2})$$

Therefore

$$f_Y(y) = \begin{cases} \frac{1}{10}(1-y^{5/2}) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

1p (b)

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y y} = \begin{cases} \frac{5x^4}{1-y^{5/2}} & \text{for } \sqrt{y} \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(Note: $f_{X|Y}(x|y)$ is considered a function of x, with y given, therefore we don't write constraints on y.)

1p (c) [Note that the "assumed" distributions are not the correct ones!]

$$\hat{X}_B = \text{E}[X] = \int_0^1 \frac{5x^5}{2} \text{d}x = \frac{5}{12}$$

1p (d)

$$\hat{X}_M(y) = \text{E}[X|y] = \int_0^1 x f_{X|Y}(x|y)\text{d}x = \int_0^1 \frac{x^3}{2(1-y^{3/2})}\text{d}x = \frac{1}{8(1-y^{3/2})}$$

[It can be observed that this cannot be the correct MMSE estimate: as $y \to 1$ then $\hat{X}(y) \to \infty$, which should not happen.]

0.5p (e) If X and Y are dependent, then knowing something about the realization of Y tells something about the realization of X. The conditional estimator (d) therefore will be "better": it has a smaller MSE (or variance).

0.5p (f) This holds if X and Y are independent, since then $f_{X|Y}(x|y) = f_X(x)$.

Question 4 (5 points)

Let Z be a random variable with a Laplace(λ) distribution:

$$f_Z(z) = \frac{\lambda}{2} e^{-\lambda|z|}, \qquad \lambda > 0$$

(a) Draw $f_Z(z)$ for $\lambda = 1$.

(b) By comparing to a Gaussian distribution, explain why a Laplace distribution is often used to model random noise with *outliers* (i.e., occasional large numbers).

(c) Derive that the moment generating function (MGF) of Z is given by

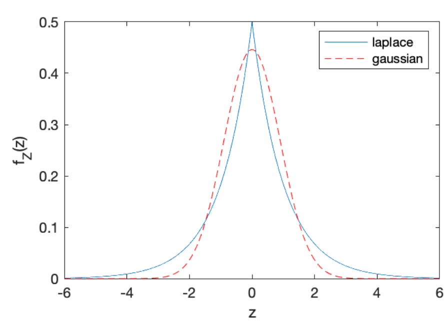
$$\phi_Z(s) = \frac{\lambda^2}{\lambda^2 - s^2}$$

and specify the ROC.

(d) Compute E[Z] and var[Z].

Solution

0.5p (a)



0.5p (b) By comparing to a Gaussian distribution, $\sim e^{-z^2}$, it is seen that large values of z are much more likely.

2p (c) The MGF is

$$\begin{aligned}
\phi_Z(z) &= \mathrm{E}[e^{sZ}] \\
&= \int_{-\infty}^{\infty} e^{sz} \frac{\lambda}{2} e^{-\lambda|z|} \mathrm{d}z \\
&= \frac{\lambda}{2} \left(\int_{-\infty}^{0} e^{z(s+\lambda)} \mathrm{d}z + \int_{0}^{\infty} e^{z(s-\lambda)} \mathrm{d}z \right) \\
&= \frac{\lambda}{2} \left(\frac{e^{z(s+\lambda)}}{s+\lambda} \Big|_{-\infty}^{0} + \frac{e^{z(s-\lambda)}}{s-\lambda} \Big|_{0}^{\infty} \right) \\
&= \frac{\lambda}{2} \left(\frac{1}{\lambda + s} + \frac{1}{\lambda - s} \right) \\
&= \frac{\lambda}{2} \left(\frac{2\lambda}{\lambda^2 - s^2} \right) \\
&= \frac{\lambda^2}{\lambda^2 - s^2}
\end{aligned}$$

which is the desired result. To make the integrals converge, we need (in line 4) $s + \lambda > 0$ resp. $s - \lambda < 0$, so the ROC is $-\lambda < s < \lambda$.

2p (d) Using the MGF

$$\mathrm{E}[Z] = \frac{\mathrm{d}}{\mathrm{d}s} \phi_Z(z) \Big|_{s=0} = \frac{2\lambda^2 s}{(\lambda^2 - s^2)^2} \Big|_{s=0} = 0$$

$$\mathrm{E}[Z^2] = \frac{\mathrm{d}^2}{\mathrm{d}s^2} \phi_Z(z) \Big|_{s=0} = \frac{2\lambda^2}{(\lambda^2 - s^2)^2} + \frac{2\lambda^2 s \cdot (-2) \cdot (-2s)}{(\lambda^2 - s^2)^3} \Big|_{s=0} = \frac{2}{\lambda^2}$$

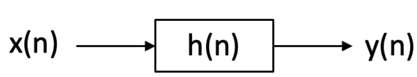
so that $\mathrm{var}[Z] = \frac{2}{\lambda^2}$.

Question 5 (6 points)

Consider an application where a signal $x[n]$ is communicated from a transmitting device to a receiving device. Signal $x[n]$ is considered to be a realization of a zero-mean WSS random process $X[n]$ with autocorrelation function

$$R_X[k] = \sigma_X^2 \left(\delta[k+1] + 3\delta[k] + \delta[k-1] \right).$$

For efficiency, prior to transmission, process $X[n]$ is first decorrelated (or whitened) by a filter with impulse response $h[n]$, leading to a white random process $Y[n]$. Let $H(\phi)$ be the DTFT of $h[n]$.



(a) Determine the power spectral density $S_X(\phi)$.

(b) Give the magnitude-squared response $|H(\phi)|^2$ of a filter that leads to a decorrelated process $Y[n]$ with variance σ_X^2.

Assume now that the impulse response $h[n]$ is given by $h[n] = \delta[n-1]$, while $R_Y[k]$ is now unknown, and $R_X[k]$ is the same as before.

(c) Give the crosscorrelation function $R_{XY}[k]$ and the autocorrelation function $R_Y[k]$ for this situation.

Suppose now that we upsample $x[n]$ by a factor $L = 2$, i.e., $v[n] = [\cdots, 0, \boxed{x[0]}, 0, x[1], 0, x[2], 0, \cdots]$. Let $V[n]$ be the corresponding random signal.

(d) Determine the autocorrelation function $R_V[n, k]$. Is $V[n]$ WSS?

Solution

1p (a)

$$S_X(\phi) = 3 + e^{-j2\pi\phi} + e^{j2\pi\phi} = 3 + 2\cos(2\pi\phi)\,.$$

1.5p (b) Decorrelated implies $R_Y[k] = \sigma_X^2 \delta[k]$, or $S_Y(\phi) = \sigma_X^2$ (constant). Since $S_Y(\phi) = |H(\phi)|^2 S_X(\phi)$, we find

$$|H(\phi)|^2 = \frac{S_Y(\phi)}{S_X(\phi)} = \frac{1}{3 + 2\cos(2\pi\phi)}\,.$$

1.5p (c)

$$R_{XY}[k] = h[k] * R_X[k] = \sigma_X^2 \left(\delta[k] + 3\delta[k-1] + \delta[k-2]\right)$$

$$R_Y[k] = h[k] * h[-k] * R_X[k] = R_X[k]\,.$$

(A delay does not change the autocorrelation function.)

2p (d) If n is even, then

$$R_V[n, k] = \mathrm{E}[X[n]X[n+k]] = \sigma_X^2 \big(\delta[k+2] + 3\delta[k] + \delta[k-2]\big)\,,$$

while if n is odd, we know that $X[n] = 0$, so

$$R_V[n, k] = \mathrm{E}[X[n]X[n+k]] = 0\,.$$

The result depends on n (even or odd), so $V[n]$ is not WSS.