Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

Partial exam EE2S31 SIGNAL PROCESSING
Part 2: 24 June 2022 (13:30-15:30)

Closed book; two sides of one A4 with handwritten notes permitted. No other tools except a basic pocket calculator permitted.

This exam consists of four questions (34 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.
Hint: Avoid losing too much time on detailed calculations, write down the general approach first.

Question 1 (8 points)

An analog signal $x_a(t)$ is a linear combination of 4 sinusoids with frequency components at 300 Hz, 400 Hz, 1.3 kHz and 4.2 kHz, as indicated in Figure 1. As depicted in Figure 2A, the signal is sampled at 2 kHz. The sampled signal is converted back to analog again using an ideal D/A converter (DAC), followed by a low-pass filter with cut-off frequency at 900 Hz. The result is $y_a(t)$.

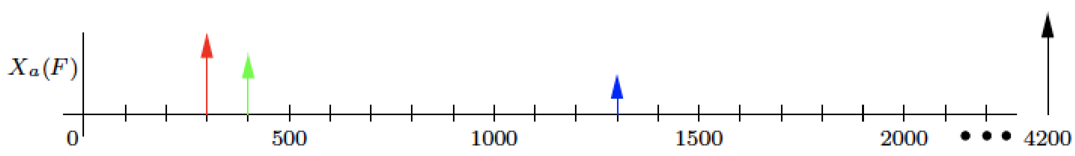


Figure 1

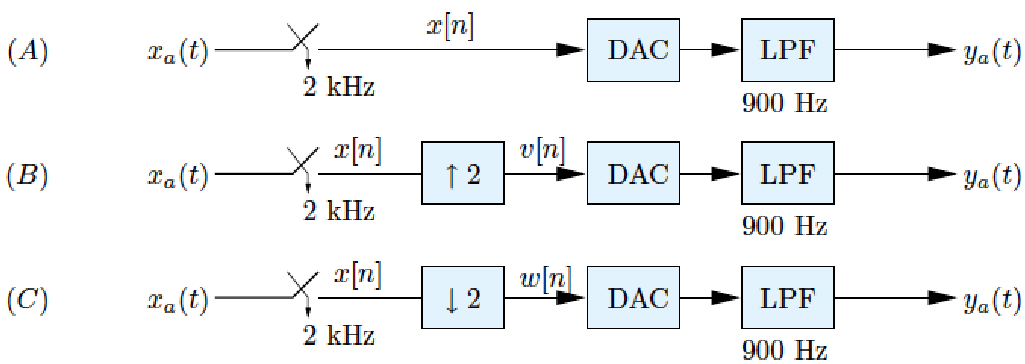


Figure 2

2 p (a) Draw the spectrum of $x[n]$ and $y_a(t)$. Where applicable, indicate both physical and normalized frequencies! What are the frequency components in the output signal?

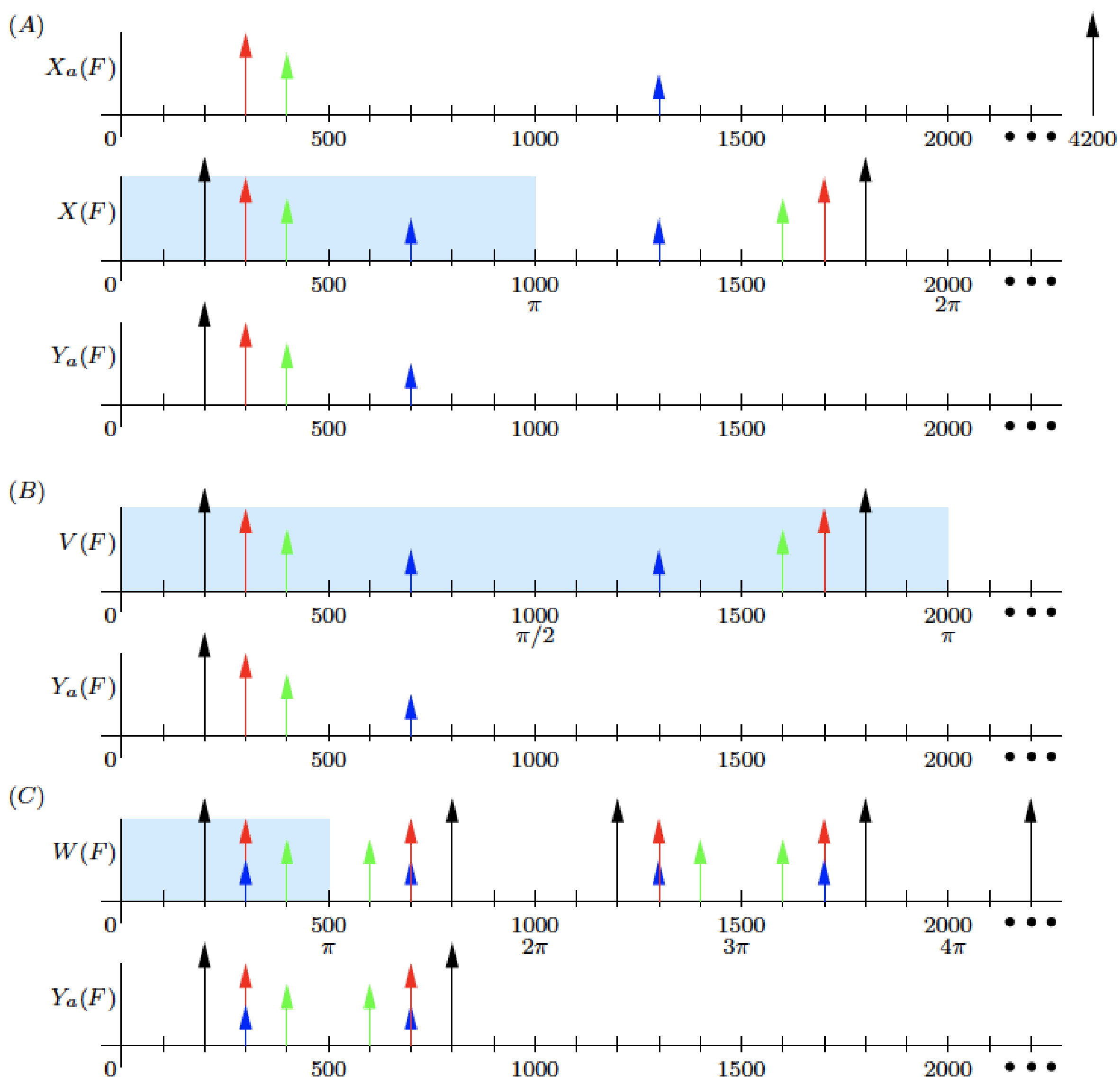
2 p (b) Now we introduce upsampling, as shown in Figure 2B. Draw the spectum of $v[n]$ and $y_a(t)$. What are the frequency components of the output now?

2 p (c) Now we introduce downsampling instead of upsampling, as shown in Figure 2C. Draw the spectum of $w[n]$ and $y_a(t)$. What are the frequency components of the output now?

2 p (d) Modify the system (C) in order to avoid aliasing. You are allowed to use an A/D converter with an arbitrary sampling rate and a digital filter. No other components in system (C) can be changed.

Solution

The resulting spectra are depicted in the Figure below.



(a) The resulting spectum is periodic at 2000 Hz and symmetric around 0. Between 0 and π (0-1000 Hz) we have:

- 300 and 400 Hz
- $1300 + k \cdot 2000$ Hz will be aliased onto 700 Hz
- $4200 + k \cdot 2000$ Hz will be aliased onto 200 Hz.

(b) The spectrum of $v[n]$ is identical with that of $x[n]$ in physical frequencies, but the link between physical and normalized frequencies changes (2π now corresponds to 4000 Hz). After the DAC and LPF we get the same output spectrum as before in part A.

(c) Compared to $X(F)$, the spectrum is streched with a factor of 2, i.e. frequencies that were filling the fundamental period $[0, \pi]$ are now stretched over $[0, 2\pi]$. Note that 2π now corresponds to 1000 Hz, i.e. the spectrum is now periodic with a period of 1000 Hz. Therefore, we have the following frequency components between $[0, 2\pi]$ (0-1000 Hz):

- 200, 300 and 400 Hz
- $700 + k \cdot 1000$ Hz will be also aliased onto 300 Hz
- Due to 2π periodicity and symmetry, we also have $-200 + 1000n = 800$ Hz, $-300 + 1000 = 700$ Hz, $-400 + 1000 = 600$ Hz.

(d) In order to avoid aliasing, we need to sample at at least twice the highest frequency in the signal, i.e 8.4 kHz. With such an an A/D converter, in our digital signal 2π corresponds to 8.4 kHz. Before downsampling with a factor of 2, we need to remove all frequencies above $\pi/2$. This means a digital low-pass filter with a cut-off at 2.1 kHz.

Question 2 (9 points)

Given a signal $x_a(t)$ with bandwidth $B = 120$ Hz and unit variance with a range between $[-1, 1]$. The signal is digitized using an A/D converter using a binary representation with 4 bits plus a sign bit.

1 p (a) What is the maximum possible value of the quantization error?

1 p (b) Assuming that the error is zero mean and is uniformly distributed, what is the power of the quantization noise?

2 p (c) The signal to quantization noise ratio is given by SQNR $= 10 \log_{10} \frac{\sigma_x^2}{\sigma_e^2}$. Show that increasing the number of bits in the A/D converter, the SQNR increases with 6 dB per bit.

2 p (d) Name two other strategies to increase the SQNR!

2 p (e) The figure below shows the frequency response of the noise ($H_e(\omega)$) and signal transfer functions ($H_s(\omega)$) of a 1st and 2nd order sigma-delta modulator (SDM). Based on this figure, explain the concept of noise shaping!

1 p (f) At which sampling rate (roughly) should we sample the signal $x_a(t)$ in order to efficiently suppress noise in the signal band?

Solution

(a) The range is $R = 2$. There are $2^{4+1} = 2^5 = 32$ possible codewords, therefore, the stepsize is $2/32 = 1/16$. The maximum quantization error is half of this (the possible maximum distance between signal and the closest quantization level), i.e. $1/32$.

(b) The power of the quantization noise is

$$P_n = \sigma_x^2 = \frac{\Delta^2}{12} = \frac{\frac{1}{16}^2}{12} = 3.25 \cdot 10^{-4}$$

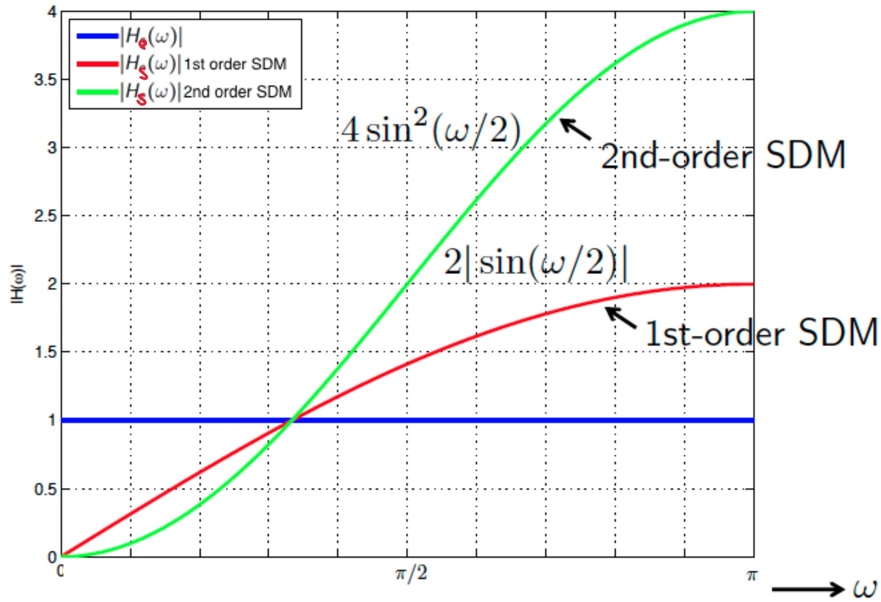


Figure 3

(c)

$$\begin{aligned}
\text{SQNR} &= 10\log_{10}\frac{\sigma_x^2}{\sigma_e^2}\\
&= 10\log_{10}\frac{\sigma_x^2}{\frac{\Delta^2}{12}}\\
&= 10\log_{10}\frac{\sigma_x^2}{\frac{(\frac{R^2}{2^{b+1}})^2}{12}}\\
&= 10\log_{10}\frac{12\sigma_x^2}{R^2} \;+\; 10\log_{10}2^{2b+2}\\
&= 10\log_{10}\frac{12\sigma_x^2}{R^2} \;+\; b\cdot 2\cdot 10\log_{10}2 \;+\; \text{const}
\end{aligned}$$

The second term is equal to $b \cdot 6$ dB, therefore, increasing b results in an improvement of 6 dB.

(d) We can use differential (or even better: differential predictive) quantization. Another strategy could be oversampling.

(e) The signal transfer function equals 1 at all frequencies, which means that it does not alter the signal. The noise transfer function, however, has high-pass characteristics, suppressing the noise at the lowest frequencies, in other words, shaping the noise such that most of its power is distributed towards the higher frequencies. This is beneficial in case the signal has low frequency.

(f) The noise is suppressed (roughly) below 0.3π, therefore, we need to choose a sampling rate F_s such that $B < \frac{F_s}{2} \cdot 0.3$. So, $F_s > 2B/(0.3) = 2 \cdot 120/0.3 = 800$ Hz.

Question 3 (8 points)

Consider the random process $X(t) = A\cos(2\pi f_0 t)$, where A is a random variable with zero mean and variance σ^2, and f_0 is a non-random frequency in Hz.

(a) Draw two realizations of $X(t)$.

(b) Determine $\mathrm{E}[X(t)]$.

(c) Compute the autocorrelation function $R_X(t,\tau)$.

(d) Is $X(t)$ a WSS random process? (Motivate)

Now, let A and B be two independent random variables with zero mean and variance σ^2, and consider $Z(t) = A\cos(2\pi f_0 t) + B\sin(2\pi f_0 t)$.

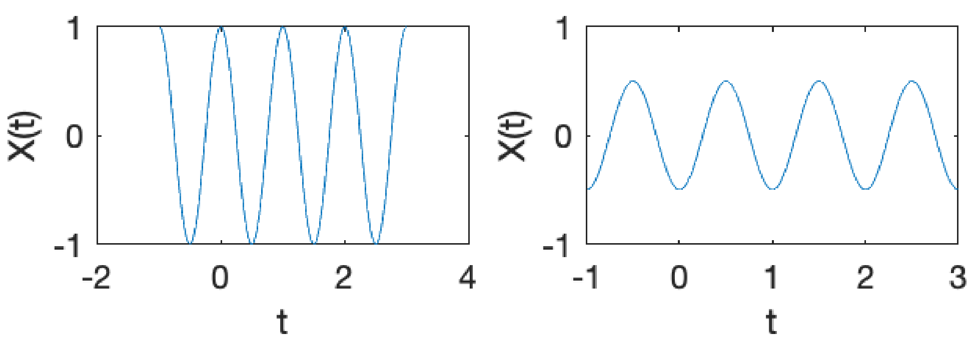
(e) Compute the autocorrelation function $R_Z(t,\tau)$.

(f) Is $Z(t)$ a WSS random process? (Motivate)

Hint: Recall $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$, $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$.

Solution

(a) 1 pnt For example, we could draw (randomly) $A = 1$ and $A = -0.5$, resulting in



(b) 1 pnt $\mathrm{E}[X(t)] = \mathrm{E}[A]\cos(2\pi f_0 t) = 0$.

(c) 2 pnt Since $\mathrm{E}[A] = 0$, we have $\mathrm{E}[A^2] = \mathrm{var}[A] + (\mathrm{E}[A])^2 = \sigma^2$.

$$\begin{aligned} R_X(t,\tau) &= \mathrm{E}[X(t)X(t+\tau)] \\ &= \mathrm{E}[A^2]\cos(2\pi f_0 t)\cos(2\pi f_0(t+\tau)) \\ &= \frac{\sigma^2}{2}[\cos(2\pi f_0\tau) + \cos(2\pi f_0(2t+\tau))] \end{aligned}$$

(d) 1 pnt Not WSS because the autocorrelation $R_X(t,\tau)$ depends on t.

(e) 2 pnt Since $\mathrm{E}[AB] = \mathrm{E}[A]\,\mathrm{E}[B] = 0$,

$$\begin{aligned} R_Z(t,\tau) &= \mathrm{E}[(A\cos(2\pi f_0 t) + B\sin(2\pi f_0 t))(A\cos(2\pi f_0(t+\tau)) + B\sin(2\pi f_0(t+\tau)))] \\ &= \mathrm{E}[A^2]\cos(2\pi f_0 t)\cos(2\pi f_0(t+\tau)) + \mathrm{E}[B^2]\sin(2\pi f_0 t)\sin(2\pi f_0(t+\tau)) + 0 \\ &= \frac{\sigma^2}{2}[\cos(2\pi f_0\tau) + \cos(2\pi f_0(2t+\tau)) + \cos(2\pi f_0\tau) - \cos(2\pi f_0(2t+\tau))] \\ &= \sigma^2\cos(2\pi f_0\tau)\,. \end{aligned}$$

(f) 1 pnt WSS because $R_Z(t,\tau)$ does not depend on t (and $\mathrm{E}[Z(t)] = 0$ does not depend on t).

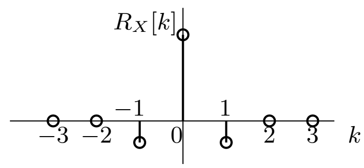
Another way to view this result is by recognizing that Z is a sinusoid with a random amplitude and phase, and the book showed that this is WSS (in contrast to a sinusoid with a non-random phase).

Question 4 (9 points)

For this question you might want to make use of Table 3, included at the end of this exam.

Consider a WSS random process $X[n]$ with autocorrelation sequence

$$R_X[k] = -\frac{1}{2}\delta[k+1] + 2\delta[k] - \frac{1}{2}\delta[k-1]$$



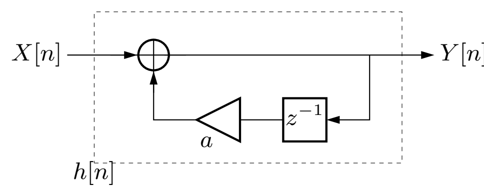
(a) Compute the power spectral density, $S_X(\phi)$, and make a plot of $S_X(\phi)$ (specify the values on the axes).

(b) Is this a valid autocorrelation sequence? (Motivate)

$X[n]$ is filtered by a first-order IIR filter with impulse response

$$h[n] = a^n u[n]\,, \qquad |a| < 1$$

resulting in the output $Y[n]$. For the moment, take $a = \frac{1}{2}$.



(c) Is $Y[n]$ an AR process? (Motivate)

(d) Find $R_{XY}[k]$, the cross-correlation sequence.

(e) Find $S_Y(\phi)$, the power spectral density of the output.

(f) Find $R_Y[k]$, the auto-correlation sequence of the output.

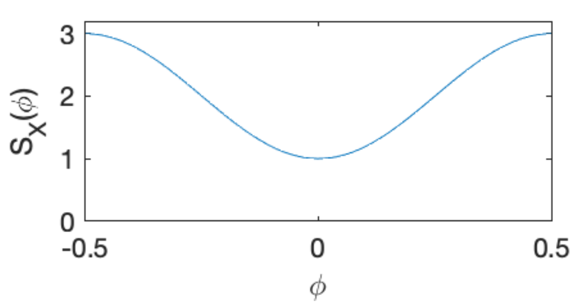
(g) Find the average power of the output.

(h) Is it possible to select a such that $Y[n]$ is white? If so, compute a.

Solution

(a) 1.5 pnt

$$S_X(\phi) = -\frac{1}{2}e^{j2\pi\phi} + 2 - \frac{1}{2}e^{-j2\pi\phi} = 2 - \cos(2\pi\phi)$$



(b) 1 pnt Yes, it is a valid autocorrelation sequence.

$R_X[k]$ satisfies the 3 properties of Thm. 13.12 ($R_X[0] \geq 0$, symmetric, maximum at $R_X[0]$), but note that these are necessary, but not sufficient conditions. We also require $S_X(\phi) \geq 0$ (which is clearly satisfied).

A counterexample showing that the 3 conditions on $\boldsymbol{R}_X[k]$ are not sufficient: consider $R_X[k] = -\frac{3}{2}\delta[k+1] + 2\delta[k] - \frac{3}{2}\delta[k-1]$. In this case we don't have $S(\phi) \geq 0$.

(c) 1 pnt Not an AR process: although the filter is AR, the input is not white. This is an ARMA process.

(d) 1 pnt

$$R_{XY}[k] = h[k] * R_X[k] = -\frac{1}{2}a^{k+1}u[k+1] + 2a^k u[k] - \frac{1}{2}a^{k-1}u[k-1]$$

with $a = \frac{1}{2}$. Inserting a, this could be written in various ways, e.g.

$$R_{XY}[k] = -\frac{1}{2}\delta[k+1] + \frac{7}{4}\delta[k] + \frac{3}{8}\left(\frac{1}{2}\right)^{k-1} u[k-1]$$

(e) 1 pnt First compute $H(z)$, from this derive $H(\phi)$ (or use Table 3):

$$H(z) = \frac{1}{1 - a^z{-1}} \qquad \Rightarrow \qquad H(\phi) = \frac{1}{1 - a^{-j2\pi\phi}}$$

$$S_Y(\phi) = H(\phi)H^*(\phi)S_X(\phi)\frac{1}{1 - a^{-j2\pi\phi}}\frac{1}{1 - a^{j2\pi\phi}}(2 - \cos(2\pi\phi)) = \frac{2 - \cos(2\pi\phi)}{(1 + a^2) - 2a\cos(2\pi\phi)}$$

with $a = \frac{1}{2}$. This simplifies to

$$S_Y(\phi) = \frac{2 - \cos(2\pi\phi)}{\frac{5}{4} - \cos(2\pi\phi)}$$

(f) 1.5 pnt Use Table 3:

$$R_Y[k] = \frac{2a^{|k|}}{1 - a^2} - \frac{\frac{1}{2}a^{|k-1|}}{1 - a^2} - \frac{\frac{1}{2}a^{|k+1|}}{1 - a^2}$$

With $a = \frac{1}{2}$ this results in

$$R_Y[k] = \frac{8}{3}\left(\frac{1}{2}\right)^{|k|} - \frac{2}{3}\left(\frac{1}{2}\right)^{|k-1|} - \frac{2}{3}\left(\frac{1}{2}\right)^{|k+1|}$$

Alternatively, write

$$S_Y(\phi) = 1 + \frac{\frac{3}{4}}{\frac{5}{4} - \cos(2\pi\phi)}$$

resulting in (using Table 3)

$$R_Y[k] = \delta[k] + \left(\frac{1}{2}\right)^{|k|}$$

(g) 1 pnt

$$R_Y[0] = \frac{2}{1 - a^2} - \frac{\frac{1}{2}a}{1 - a^2} - \frac{\frac{1}{2}a}{1 - a^2} = \frac{2 - a}{1 - a^2} = 2$$

(h) 1 pnt Yes: we need $R_Y[k]$ to be a delta spike, or $S_Y(\phi)$ to be constant; the latter requires

$$\frac{2a}{1 + a^2} = \frac{1}{2}$$
$$a^2 - 4a + 1 = 0 \qquad \Rightarrow \qquad a = 2 \pm \sqrt{3}$$

For stability reasons, we select $a = 2 - \sqrt{3} = 0.268$.