Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

Resit exam EE2S31 SIGNAL PROCESSING
20 July 2022 (13:30-16:30)

Closed book; two sides of one A4 with handwritten notes permitted. No other tools except a basic pocket calculator permitted.

This exam consists of five questions (36 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.
Hint: Avoid losing too much time on detailed calculations, write down the general approach first.

Question 1 (7 points)

Let Y be a random variable with PDF

$$f_Y(y) = \begin{cases} c(y+1) & -1 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

where c is a constant. Given $Y = y$, the random variable X is uniformly distributed between -1 and y.

(a) Find the constant c.

(b) Find the conditional PDF, $f_{X|Y}(x|y)$.

(c) Find the joint PDF, $f_{X,Y}(x,y)$, and make a diagram that shows the support of $f_{X,Y}(x,y)$.

(d) Find $\hat{X}_M(Y)$, the minimum mean square error (MMSE) estimator for X given a single sample of Y.

(e) Find $\hat{Y}_{\text{ML}}(X)$, the maximum likelihood estimator for Y given a single sample of X.

Solution

1p (a)

$$\begin{aligned} \int_{-\infty}^{\infty} f_Y(y)\,dy &= \int_{-1}^{1} c(1+y)\,dy \\ &= c\left[y + \frac{1}{2}y^2\right]_{-1}^{1} \\ &= 2c \end{aligned}$$

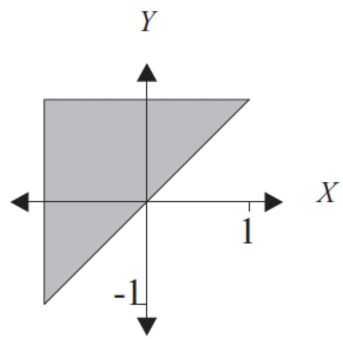
Hence $c = \frac{1}{2}$.

1p (b) Directly from the description,

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{1}{y+1} & -1 \le x \le y \\ 0 & \text{otherwise} \end{cases}$$

1p (c) The joint PDF is

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)\, f_Y(y) = \begin{cases} c & -1 \le x \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$



2p (d) The MMSE is $\hat{X}_M(Y) = \mathrm{E}[X|Y]$. Using $f_{X|Y}(x|y)$,

$$\begin{aligned} \mathrm{E}[X|Y=y] &= \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,\mathrm{d}x \\ &= \int_{-1}^{y} x \frac{1}{1+y}\,\mathrm{d}x \\ &= \frac{1}{1+y}\left[\frac{1}{2}x^2\right]_{x=-1}^{y} \\ &= \frac{1}{1+y}\left(\frac{1}{2}y^2 - \frac{1}{2}\right) \\ &= \frac{1}{2}(y-1) \end{aligned}$$

Then $\hat{X}_M(Y) = \mathrm{E}[X|Y] = \frac{1}{2}(Y-1)$.

2p (e) By definition, $\hat{y}_{\mathrm{ML}}(x) = \arg\max_y f_{X|Y}(x|y)$. Then

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y+1} & -1 \le x \le y \\ 0 & \text{otherwise} \end{cases}$$

which should be seen as a function on y. For any x, this function has a support on $x \le y \le 1$.

$$\underset{y, y \ge x}{\arg\max} \frac{1}{y+1} = x$$

Hence, $\hat{Y}_{\mathrm{ML}}(X) = X$.

Question 2 (6 points)

For this question you might want to make use of Table 4.2, included at the end of this exam.

Random variables $X_1, X_2, \cdots$ are iid, with mean value $\mu = 75$ and standard deviation $\sigma = 15$. Let

$$M_n(X) = \frac{X_1 + \cdots + X_n}{n}$$

be the sample mean.

(a) How many samples n do we need to guarantee that $M_n(X)$ is between 74 and 76 with probability larger than 0.99?

[Hint: use Chebyshev]

(b) If we also know that each X_i is Gaussian distributed, then how many samples n would we need to expect that $M_n(X)$ is between 74 and 76 with probability larger than 0.99?

[Hint: use the CLT]

Suppose now that the X_i are not iid but correlated, such that

$$\begin{cases} \text{var}[X_i] = \sigma^2 & i = 1, \cdots, n \\ \text{cov}[X_i, X_{i+1}] = \sigma^2 a & i = 1, \cdots, n-1 \\ \text{cov}[X_i, X_j] = 0 & \text{otherwise} \end{cases} \tag{1}$$

where a is a constant such that $|a| < 1$.

(c) Show that $\text{var}[X_1 + \cdots + X_n] = \sigma^2[n + 2(n-1)a]$.

(d) Let $a = 0.8$. Repeat part (a): How many samples n do we now need to guarantee that $M_n(X)$ is between 74 and 76 with probability larger than 0.99?

Solution

1.5p (a) We would like to find the value of n such that

$$\text{P}[74 \leq M_n(X) \leq 76] \geq 0.99$$

Note $\text{var}[M_n(X)] = \text{var}[X]/n = \sigma^2/n$. Using the Chebyshev inequality:

$$\begin{aligned} \text{P}[74 \leq M_n(X) \leq 76] &= 1 - \text{P}[\,|M_n(X) - \text{E}[X]| \geq 1] \\ &\geq 1 - \frac{\text{var}[X]}{n} = 1 - \frac{255}{n} \geq 0.99 \end{aligned}$$

This yields $n \geq 22,500$.

2p (b) If each X_i is a Gaussian, the sample mean $M_n(X)$ will also be Gaussian with mean and variance

$$\begin{aligned} \mu &= \text{E}[M_n(X)] = \text{E}[X] = 75\,, \\ \sigma^2 &= \text{var}[M_n(X)] = \frac{\text{var}[X]}{n} = \frac{225}{n} \end{aligned}$$

In this case,

$$\begin{aligned} \text{P}[74 \leq M_n(X) \leq 76] &= \text{P}\left[\frac{-1}{\sigma} \leq \frac{M_n(X) - \mu}{\sigma} \leq \frac{1}{\sigma}\right] \\ &= \Phi\left(\frac{1}{\sigma}\right) - \Phi\left(\frac{-1}{\sigma}\right) \\ &= 2\Phi\left(\frac{1}{\sigma}\right) - 1 \\ &= 2\Phi\left(\frac{\sqrt{n}}{15}\right) - 1 \\ &\geq 0.99 \end{aligned}$$

Thus $\Phi(\frac{\sqrt{n}}{15}) \geq 0.995$. Using Table 4.2, we find $\frac{\sqrt{n}}{15} \geq 2.58$ so that $n \geq 1,498$.

1p (c) Use Thm 9.2: the variance of $S = X_1 + \cdots + X_n$ is

$$\mathrm{var}[S] = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{cov}[X_i, X_j] = \sum_{i=1}^{n} \mathrm{var}[X_i] + 2\sum_{i=1}^{n-1}\sum_{j=2}^{n} \mathrm{cov}[X_i, X_j]$$

This yields the result.

1.5p (d) Since the variance is higher we expect to need more samples.

Using the Chebyshev inequality:

$$\mathrm{P}[74 \le M_n(X) \le 76] \;=\; \mathrm{P}[\,|S - \mathrm{E}[S]| \le n] \;\ge\; 1 - \frac{\mathrm{var}[S]}{n^2} \;\ge\; 0.99$$

$$\frac{\sigma^2[n + 2(n-1)a]}{n^2} \le 0.01$$

Since n will be large, we ignore the term $2a$ for simplicity.

$$\frac{\sigma^2[n + 2an]}{n^2} \le 0.01$$

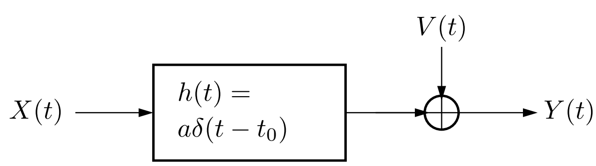
Inserting $\sigma = 15$ and $a = 0.8$ gives

$$n \ge \frac{225 \cdot 2.6}{0.01} = 58,500$$

Question 3 (6 points)

For this question you might want to make use of Table 1,2, included at the end of this exam.

In EPO4, we use an audio beacon to estimate a propagation delay. In this question we consider a simple model for this.



Let the beacon signal $X(t)$ be a zero mean wide-sense stationary (WSS) signal, with auto-correlation function

$$R_X(\tau) = \frac{\sin(2000\pi\tau)}{2000\pi\tau}\,.$$

Let $V(t)$ be white Gaussian noise (independent of $X(t)$) with variance σ_V^2. The audio channel is simply modeled with an attenuation a and a delay t_0. The received signal is

$$Y(t) = aX(t - t_0) + V(t)$$

Thus, the channel is represented by a filter with impulse reponse $h(t) = a\delta(t - t_0)$.

(a) Determine the auto-correlation function $R_Y(t, \tau)$.

(b) Determine the cross-correlation function $R_{XY}(t, \tau)$.

(c) Is $Y(t)$ WSS?

(d) Are $X(t), Y(t)$ jointly WSS?

(e) Determine the input power spectral density $S_X(f)$.

(f) Determine the output power spectral density $S_Y(f)$ and the cross power spectral density $S_{XY}(f)$.

(g) Given (estimates of) $R_X(\tau)$, $R_{XY}(t,\tau)$ and $R_Y(t,\tau)$, how can we estimate t_0?

Solution

1.5p (a) $V(t)$ has auto-correlation function $R_V(\tau) = \sigma_V^2 \delta(t-\tau)$.

$$\begin{aligned} R_Y(t,\tau) &= \mathrm{E}[Y(t)Y(t+\tau)] \\ &= \mathrm{E}[(aX(t-t_0)+V(t))(aX(t-t_0+\tau)+V(t+\tau))] \\ &= a^2 R_X(\tau) + R_V(\tau) \\ &= a^2 \frac{\sin(2000\pi\tau)}{2000\pi\tau} + \sigma_V^2\delta(\tau) \end{aligned}$$

1p (b)

$$\begin{aligned} R_{XY}(t,\tau) &= \mathrm{E}[X(t)Y(t+\tau)] \\ &= \mathrm{E}[X(t)(aX(t-t_0+\tau)+V(t+\tau))] \\ &= aR_X(\tau-t_0) \\ &= a\frac{\sin(2000\pi(\tau-t_0))}{2000\pi(\tau-t_0)} \end{aligned}$$

0.5p (c) We have already verified that $R_Y(t,\tau)$ depends only on the time difference. Since $\mathrm{E}[Y(t)] = a\mathrm{E}[X(t-t_0)] = a\mu_X + \mu_V = 0$, thus is a constant independent of t, we have verified that $Y(t)$ is wide sense stationary.

0.5p (d) Since $X(t)$ and $Y(t)$ are both wide sense stationary and since we have shown that $R_{XY}(t,\tau)$ depends only on τ, we know that $X(t)$ and $Y(t)$ are jointly wide sense stationary.

0.5p (e) Using Table 1,

$$S_X(f) = \frac{1}{2000}\mathrm{rect}\left(\frac{f}{2000}\right)$$

1p (f) $H(f) = ae^{-j2\pi f t_0}$, so that

$$S_Y(f) = |H(f)|^2 S_X(f) + S_V(f) = a^2 S_X(f) + \sigma_V^2$$

$$S_{XY}(f) = ae^{-j2\pi f t_0} S_X(f)$$

Alternatively, compute the Fourier Transform of the answers under items (a), (b).

1p (g) From $R_X(\tau)$ and $R_{XY}(t,\tau)$, compute the PSDs $S_X(f)$ and $S_{XY}(f)$, and an estimate for $H(f)$ as

$$\hat{H}(f) = \frac{S_{XY}(f)}{S_X(f)}$$

and since $H(f) = ae^{-j2\pi f t_0}$, we find t_0 from the argument of the estimate of $H(f)$ (or: take a logarithm and then take the imaginary part). More in detail, we will need to fit a line through $2\pi f t_0$ as function of f.

Alternatively, from the estimate of $H(f)$, compute the corresponding impulse response $h(t)$, and then search for the peak. This is more robust in case the channel is not exactly a delta spike.

Question 4 (10 points)

2p (a) Write down the 4-point DFT matrix, and explain how the elements of the matrix are related to the fourth roots of unity using a sketch on the unit circle.

2p (b) Show (using any method you prefer) that the DFT of $x[n] = \begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}$ is $X[k] = \begin{bmatrix}10 & -2+2j & -2 & -2-2j\end{bmatrix}$.

2p (c) Compute the 8-point DFT of the sequence $y[n] = \begin{bmatrix}1 & 0 & 2 & 0 & 3 & 0 & 4 & 0\end{bmatrix}$ using a linear combination of two 4-point DFTs.

2p (d) The 8-point decimation-in-time algorithm is illustrated in a butterfly diagram in Figure 1 on the Answer Sheet, included at the end of this exam.

Use now this diagram to compute the 8-point DFT of $y[n]$. Specifically, indicate all intermediate results in the signal flow graph and hand it in as your solution.

2p (e) How many butterflies would the radix-2 FFT algorithm have for the computation of a 32-point DFT?

Solution

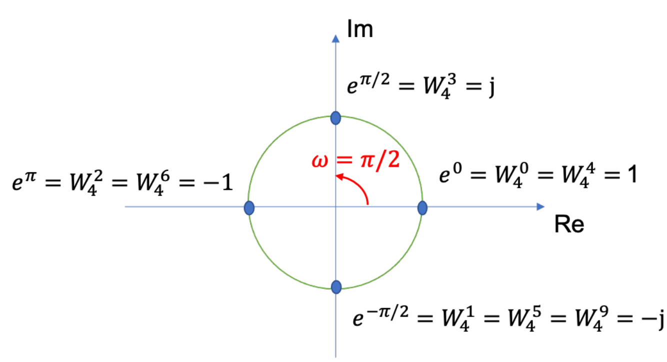
(a) Let $W_N = e^{-j2\pi/N}$. Therefore, by definition, the N-point DFT matrix is an N-by-N matrix where the entry in row n and column k is $W_N^{n\cdot k}$. Then, the 4-point DFT matrix is

$$\begin{bmatrix} W_4^{0\cdot0} & W_4^{0\cdot1} & W_4^{0\cdot2} & W_4^{0\cdot3} \\ W_4^{1\cdot0} & W_4^{1\cdot1} & W_4^{1\cdot2} & W_4^{1\cdot3} \\ W_4^{2\cdot0} & W_4^{2\cdot1} & W_4^{2\cdot2} & W_4^{2\cdot3} \\ W_4^{3\cdot0} & W_4^{3\cdot1} & W_4^{3\cdot2} & W_4^{3\cdot3} \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}$$

Using the fact that $W_4 = e^{-j\pi/2}$, we see that

$$\begin{aligned} W_4^0 &= e^{-j\pi/2\cdot0} = e^{j0} = 1 \\ W_4^1 &= e^{-j\pi/2\cdot1} = e^{-j\pi/2} = -j \\ W_4^2 &= e^{-j\pi/2\cdot2} = e^{-j\pi} = -1 \\ W_4^3 &= e^{-j\pi/2\cdot3} = e^{-j3\pi/2} = j \end{aligned}$$

These are the 4 roots of unity (i.e., equidistant points lying on the unit circle). Higher powers of W_N fall on the same points (periodicity!), as shown on the plot below.



(b) The DTF can be computed using the DFT matrix:

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

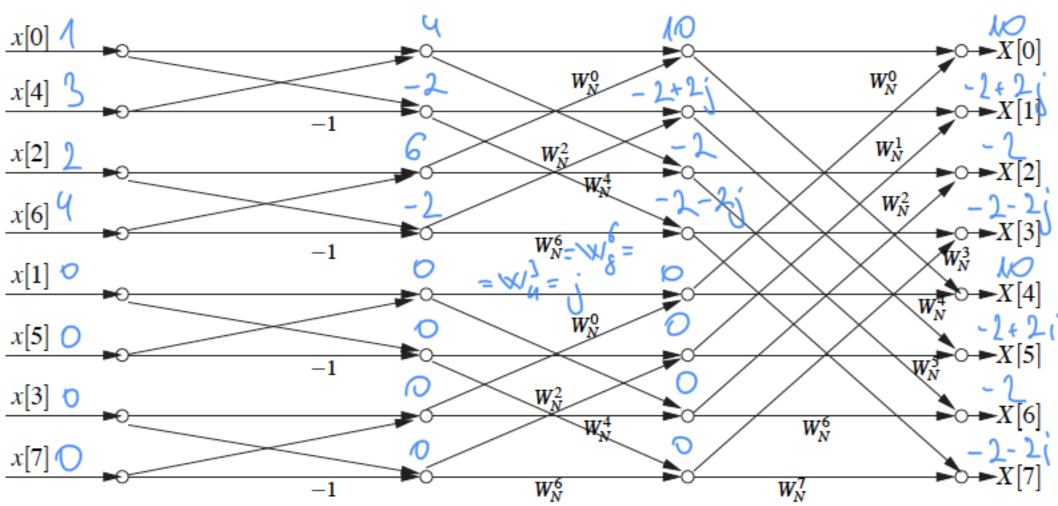
(c) The 8-point DFT $X[k]$ of $x[k]$ can be calculated from combining the 4-point DFTs of the even and the odd samples as follows:

$$\begin{aligned} X[k] &= G[k] + W_N^k H[k], \qquad k = 0, 1, \cdots, N/2-1 \\ X[k+N/2] &= G[k] - W_N^k H[k], \qquad k = 0, 1, \cdots, N/2-1 \end{aligned}$$

with $G[k]$ being the 4-point DFT of the even samples and $H[k]$ being the DFT of the odd samples. The odd samples are all 0s, so $H[k] = 0$. We know $G[k]$ from the previous subquestion. Therefore,

$$X[k] = \begin{bmatrix} 10 & -2+2j & -2 & -2-2j & 10 & -2+2j & -2 & -2-2j \end{bmatrix}$$

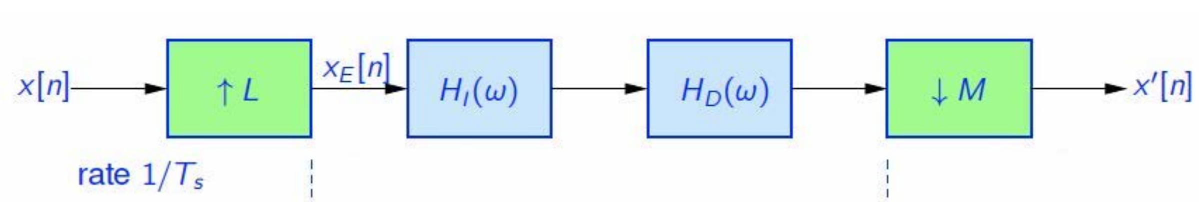
(d)



(e) There are $\log_2(32) = 5$ stages, and $N/2 = 32/2 = 16$ butterflies per stage, i.e., 80 butterflies.

Question 5 (7 points)

Given the following conversion rate system with $L = 3$, $M = 2$, and $T_s = 10$ms.



1p (a) What are the data rates of the signals $x[n]$, $x_E[n]$ and $x'[n]$?

2p (b) What are the roles of the filters $H_I(\omega)$ and $H_D(\omega)$?

2p (c) Can you unite these filters into one filter? If yes, give the specification of the resulting single (ideal) filter in terms of its cut-off frequency. If not, explain why!

2p (d) Develop an alternative implementation of this system, where the filter(s) run at 100Hz data rate, using a polyphase representation and a noble identity. Make two sketches! In the first sketch, draw the modified system after using the polyphase representation. In the second sketch, draw the system after applying the appropriate noble identity as well!

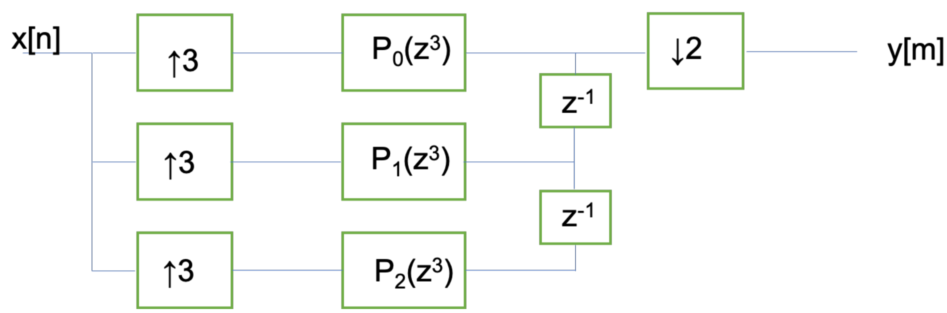
Solution

(a) The data rates are the following: $x[n] \to$ 100 Hz, $x_E[n] \to$ 300 Hz , $x'[n] \to$ 150 Hz.

(b) $H_l(\omega)$ is a low-pass filter to reject the extra copies of the spectral image from the fundamental interval. $H_D(\omega)$ is an antialiasing (low-pass) filter.

(c) Yes, they can be united. The cut-off of the single, united low-pass filter is at $\min(\pi/L, \pi/M) = \min(\pi/3, \pi/2) = \pi/3 \leftrightarrow 150/3$ Hz. (Note, the data rate is $2\pi \leftrightarrow 300$ Hz.)

(d) In order to run at 100 Hz, we need to switch the order of the filter and the upsampler ($L = 3$). For this, we first need to use the 3-phase polyphase representation of the (united) filter:



Now, we can swap the filter and the upsampler: