The entire exam had 50 points. The grading was such that 37 points corresponded to grade 10.0

Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

Partial exam EE2S31 SIGNAL PROCESSING
Part 1: May 28, 2021
Block 1: Stochastic Processes (13:30-14:30)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted.
Upload answers during 14:25–14:40

This block consists of three questions (25 points); more than usual, and this will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.
Hint: Avoid losing too much time on detailed calculations, write down the general approach first.

Question 1 (9 points)

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c\,xy & \text{for} \quad 0 \le x \le 1,\ 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant c.

(b) Find $f_X(x)$ and $f_Y(y)$.

(c) Are X and Y independent?

(d) Determine $\mathrm{P}[X + Y \le 1]$.

(e) Determine $\mathrm{E}[X|X + Y \le 1]$.

Solution

(a) 2 pnt

$$\begin{aligned} \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x &= \int_{x=0}^{1} \int_{y=0}^{1} c\,xy\,\mathrm{d}y\,\mathrm{d}x \\ &= c \int_0^1 \left[\frac{1}{2} xy^2\right]_{y=0}^{1} \mathrm{d}x \\ &= \frac{c}{2} \int_0^1 x\,\mathrm{d}x \\ &= \frac{c}{4} = 1 \end{aligned}$$

Hence $c = 4$.

(b) 2 pnt For $0 \le x \le 1$,

$$f_X(x) = \int_{y=0}^{x} c\,xy\,\mathrm{d}y = c\,x \left[\frac{1}{2} y^2\right]_0^1 = 2x$$

The complete PDF is

$$f_X(x) = \begin{cases} 2x & \text{for} \quad 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Based on symmetry, $f_Y(y) = f_X(y)$, hence

$$f_Y(y) = \begin{cases} 2y & \text{for} \quad 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(c) 1 pnt Clearly $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ so X and Y are independent.

(d) 1.5 pnt

$$\begin{aligned} \mathrm{P}[X+Y \le 1] &= \iint_{x+y\le 1} f_{X,Y}(x,y)\,\mathrm{d}x\mathrm{d}y \\ &= \int_0^1 \int_0^{1-x} 4xy\,\mathrm{d}y\mathrm{d}x \\ &= \int_0^1 2x(1-x)^2\,\mathrm{d}x \\ &= 2\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{1}{6} \end{aligned}$$

Alternatively (but less insightful): define $W = X+Y$, then

$$f_W(w) = \int_0^w f_{X,W-X}\mathrm{d}x = \int_0^w 4x(w-x)\mathrm{d}x = 4\left[\frac{1}{2}wx^2 - \frac{1}{3}x^3\right]_0^w = \frac{2}{3}w^3\,, \quad (0 \le w \le 2)$$

$$\mathrm{P}[X+Y \le 1] = \int_0^1 f_W(w)\mathrm{d}w = \int_0^1 \frac{2}{3}w^3\mathrm{d}w = \frac{2}{3}\frac{1}{4} = \frac{1}{6}$$

(e) 2.5 pnt The conditional PDF is

$$f_{X,Y|X+Y\le 1}(x,y) = \frac{f_{X,Y}(x,y)}{\mathrm{P}[X+Y \le 1]}$$

on the domain of the constraint, and 0 otherwise:

$$f_{X,Y|X+Y\le 1}(x,y) = \begin{cases} 24\,xy & \text{for} \quad 0 \le x \le 1,\ 0 \le y \le 1-x \\ 0 & \text{otherwise.} \end{cases}$$

Then (cf. Thm. 7.7)

$$\begin{aligned} \mathrm{E}[X|X+Y \le 1] &= \iint x\, f_{X,Y|X+Y\le 1}(x,y)\,\mathrm{d}x\mathrm{d}y \\ &= 24\int_0^1 \int_0^{1-x} x^2 y\,\mathrm{d}y\mathrm{d}x \\ &= 12\int_0^1 x^2(1-x)^2\,\mathrm{d}x \\ &= 12\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) = \frac{2}{5} \end{aligned}$$

Alternatively: compute

$$f_{X|X+Y\le 1}(x) = \int_0^{1-x} 24xy\mathrm{d}y = 12x(1-x)^2 = 12(x-2x^2+x^3) \quad (0 \le x \le 1)$$

$$\mathrm{E}[X|X+Y\le 1] = \int x\, f_{X|X+Y\le 1}(x)\mathrm{d}x = 12\int_0^1 (x^2-2x^3+x^4)\mathrm{d}x = 12(\frac{1}{3}-\frac{2}{4}+\frac{1}{5}) = \frac{2}{5}$$

Question 2 (7 points)

It is known that if X is standard normal distributed, $X \sim \text{Gaussian}(0,1)$, then $Y = X^2$ is Chi-square distributed with 1 degree of freedom.

Further, it is known that if Y has a Chi-square distribution with n degrees of freedom, the moment generating function (MGF) is given by

$$\phi_Y(s) = \frac{1}{(1-2s)^{n/2}}\,, \qquad \text{ROC: } s < \frac{1}{2}$$

(a) Show that if $X_i \sim \text{Gaussian}(0,1)$ (all independent), then $Y = \sum_1^n X_i^2$ has a Chi-squared distribution with n degrees of freedom.

(b) Use the MGF to prove that

$$\mathrm{E}[Y] = n, \qquad \mathrm{var}[Y] = 2n\,.$$

Suppose now we have n iid random variables $X_i \sim \text{Gaussian}(0,\sigma)$, and we try to estimate the variance σ^2 using

$$S_n = \frac{1}{n}\sum_{i=1}^n X_i^2$$

(c) What is $\mathrm{E}[S_n]$ and $\mathrm{var}[S_n]$?

Is the estimate S_n unbiased? Is it consistent?

(d) Use the central limit theorem to estimate how many samples n are at least needed such that

$$\mathrm{P}[\,|S_n - \sigma^2| > 0.1\sigma^2] \;<\; 0.01$$

Note: You will need to use table 4.1/4.2 on p. 129/130.

Solution

(a) 1 pnt For $n = 1$, we have $Y_1 = X_1^2$ and know that

$$\phi_{Y_1}(s) = \frac{1}{(1-2s)^{1/2}}$$

Then for the sum of n iid random variables of this form, Thm. 9.6 gives that

$$\phi_Y(s) = [\phi_{Y_1}(s)]^n = \frac{1}{(1-2s)^{n/2}}$$

which is the MGF of a Chi-square distribution with n degrees of freedom.

(b) 2 pnt Using the MGF,

$$\begin{aligned}
\mathrm{E}[Y] &= \left.\frac{\mathrm{d}\phi_Y(s)}{\mathrm{d}s}\right|_{s=0} = \left.\frac{n/2\cdot 2}{(1-2s)^{n/2+1}}\right|_{s=0} = n \\
\mathrm{E}[Y^2] &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}\phi_Y(s)}{\mathrm{d}s}\right|_{s=0} = \left.\frac{\mathrm{d}}{\mathrm{d}s}\frac{n}{(1-2s)^{n/2+1}}\right|_{s=0} = \left.\frac{n(n/2+1)\cdot 2}{(1-2s)^{n/2+2}}\right|_{s=0} \\
&= n(n+2) \\
\mathrm{var}[Y] &= \mathrm{E}[Y^2] - (\mathrm{E}[Y])^2 = n(n+2) - n^2 = 2n
\end{aligned}$$

(c) 2 pnt In comparison to Y we had before, we have $S_n = \frac{\sigma^2}{n}Y$. Thus, the mean value scales with σ^2/n and the variance with σ^4/n^2:

$$\mathrm{E}[S_n] = \sigma^2\,, \qquad \mathrm{var}[S_n] = \frac{2\sigma^4}{n}$$

The estimate is unbiased (expected value equal to the true value), and consistent (variance goes to zero for $n \to \infty$ so that the estimate converges to the true value).

(d) 2 pnt The mean value of S_n is σ^2 and its standard deviation is $\mathrm{std} = \frac{\sqrt{2}}{\sqrt{n}}\sigma^2$. To use the CLT, we need to normalize S_n as

$$Z_n = \frac{S_n - \sigma^2}{\mathrm{std}}$$

So

$$\begin{aligned}
\mathrm{P}[\,|S_n - \sigma^2| < 0.1\sigma^2] &= \mathrm{P}\left[-0.1\frac{\sigma^2}{\mathrm{std}} < Z_n < 0.1\frac{\sigma^2}{\mathrm{std}}\right] \\
&= \mathrm{P}\left[-0.1\frac{\sqrt{n}}{\sqrt{2}} < Z_n < 0.1\frac{\sqrt{n}}{\sqrt{2}}\right] \;<\; 0.99 \\
\mathrm{P}\left[Z_n < 0.1\frac{\sqrt{n}}{\sqrt{2}}\right] &< 0.995 \\
\Phi\left(0.1\frac{\sqrt{n}}{\sqrt{2}}\right) &> 0.995 \\
0.1\frac{\sqrt{n}}{\sqrt{2}} &> 2.58 \quad \text{(using table)} \\
n &> (2.58)^2\,2/(0.1)^2 = 1331
\end{aligned}$$

Question 3 (9 points)

In a BPSK communication system, a source wishes to communicate a random bit X to a receiver. The possible bits $X = 1$ and $X = -1$ are equally likely. In this system, the source transmits X multiple times. In the ith transmission, the receiver observes $Y_i = X + N_i$. After n transmissions of X, the receiver has observed $\boldsymbol{Y} = \boldsymbol{y} = [y_1, \cdots, y_n]^T$.

Assume the noise N_i are iid Gaussian(0,1) random variables, independent of X.

Let $\hat{X}_{\mathrm{ML}}(\boldsymbol{y})$ be the maximum likelihood (ML) estimate of X based on the observation $\boldsymbol{Y} = \boldsymbol{y}$.

(a) Show that

$$f_{\boldsymbol{Y}|X}(\boldsymbol{y}|x) = c\,e^{-\frac{1}{2}\sum_{i=1}^{n}(y_i - x)^2}\,,$$

where c is some constant.

(b) What is $\hat{X}_{\mathrm{ML}}(\boldsymbol{y})$?

(c) Is knowledge that $X \in \{1, -1\}$ used by the ML estimator? Does the noise variance play a role?

(d) Compute e_{ML}, the mean square error of the ML estimate.

Let $\hat{X}_L(\boldsymbol{y}) = \boldsymbol{a}^T \boldsymbol{y} + b$ be the linear minimum mean square error (LMMSE) estimate of X.

(e) Find the LMMSE estimate $\hat{X}_L(\boldsymbol{y})$.

(f) What is e_L, the mean square error of the optimum linear estimate.

Hint: for (e), you may want to exploit Woodbury's Identity,

$$(\boldsymbol{I} + \boldsymbol{u}\boldsymbol{u}^T)^{-1} = \boldsymbol{I} - \frac{\boldsymbol{u}\boldsymbol{u}^T}{1 + \boldsymbol{u}^T \boldsymbol{u}}$$

Solution

(a) 1 pnt If $X = x$ is known, then $Y_i = x + N_i$, hence the PDF of Y_i is the PDF of N_i, with the mean shifted by x. Using the iid property, it follows for N samples that

$$\begin{aligned} f_{\boldsymbol{Y}|X}(\boldsymbol{y}|x) &= \prod_{i=1}^{n} f_{Y_i|X}(y_i|x) \\ &= \prod_{i=1}^{n} f_{N_i}(y_i - x) \\ &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-(y_i - x)^2/2} \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}(y_i - x)^2} . \end{aligned}$$

(b) 2 pnt The maximum likelihood estimator is obtained for the maximum of the conditional PDF, seen as function of X: it is

$$\hat{x} = \arg\max_x f_{\boldsymbol{Y}|X}(\boldsymbol{y}|x) = \arg\min_x \sum_{i=1}^{n} (y_i - x)^2$$

To obtain the minimum, set the derivative to zero:

$$\frac{\mathrm{d}}{\mathrm{d}x} \sum_{i=1}^{n} (y_i - x)^2 = -2 \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} 2x = 0$$

Hence

$$\hat{x} = \frac{1}{n} \sum_{i=1}^{n} y_i$$

(c) 1 pnt The prior information on X is by definition not used by the ML, as is also clear from the above derivation. The noise variance σ_n^2 also doesn't play a role in the resulting estimator: it is independent of the value of σ_n^2.

(d) 1.5 pnt The MSE of the ML estimator is

$$e_{\text{ML}} = \text{E}[(\hat{X} - X)^2] = \text{E}[(\frac{1}{n}\sum Y_i - X)^2] = \frac{1}{n^2}\sum \text{E}[N_i^2] = \frac{1}{n}\,.$$

(e) 2 pnt Note that X is a scalar, and $\boldsymbol{Y}$ is a vector. For the LMMSE estimator, we will need $\mu_X, \boldsymbol{\mu_Y}, \boldsymbol{C_{XY}}, \boldsymbol{C_Y}$.

$$\begin{aligned}
\mu_X &= 0, \qquad \boldsymbol{\mu_Y} = \mathbf{0} \\
C_{XY_i} &= \text{E}[XY_i] = \text{E}[X(X+N_i)] = \text{E}[X^2] = 1 \\
\Rightarrow \quad \boldsymbol{C_{XY}} &= [1, \cdots, 1] = \mathbf{1}^T \\
C_{Y_iY_j} &= \text{E}[Y_iY_j] = \text{E}[(X+N_i)(X+N_j)] = \text{E}[X^2] + \text{E}[N_iN_j] = 1 + \delta_{ij} \\
\Rightarrow \quad \boldsymbol{C_Y} &= \boldsymbol{I} + \mathbf{1}\mathbf{1}^T
\end{aligned}$$

Using Woodbury's identity, and $\mathbf{1}^T\mathbf{1} = n$,

$$\begin{aligned}
\boldsymbol{C_Y}^{-1} &= \boldsymbol{I} - \frac{\mathbf{1}\mathbf{1}^T}{1 + \mathbf{1}^T\mathbf{1}} = \boldsymbol{I} - \frac{1}{n+1}\mathbf{1}\mathbf{1}^T\,. \\
\boldsymbol{a}^T &= \boldsymbol{C_{XY}}\boldsymbol{C_Y}^{-1} = \mathbf{1}^T(\boldsymbol{I} - \frac{1}{n+1}\mathbf{1}\mathbf{1}^T) = \mathbf{1}^T - \frac{n}{n+1}\mathbf{1}^T = \frac{1}{n+1}\mathbf{1}^T
\end{aligned}$$

The LMMSE estimator is

$$\hat{X}_L(\boldsymbol{y}) = \boldsymbol{a}^T\boldsymbol{y} + b = \frac{1}{n+1}\mathbf{1}^T\boldsymbol{y} + 0 = \frac{1}{n+1}\sum_{i=1}^{n} y_i$$

(f) 1.5 pnt The MSE of this estimator is

$$\begin{aligned}
e_L &= \text{E}[(\hat{X} - L - X)^2] \\
&= \text{E}\left[\left((\frac{1}{n+1}\sum Y_i) - X\right)^2\right] \\
&= \text{E}\left[\left(\frac{1}{n+1}(\sum Y_i - X) - \frac{1}{n+1}X\right)^2\right] \\
&= \frac{1}{(n+1)^2}\text{E}\left[\left((\sum N_i) - X\right)^2\right] \\
&= \frac{1}{(n+1)^2}\left(\sum \text{E}[N_i^2] + \text{E}[X^2]\right) \\
&= \frac{1}{(n+1)^2}(n+1) \\
&= \frac{1}{n+1}\,.
\end{aligned}$$

This is lower than the MSE of the ML estimator, but the estimator is slightly biased: if $X = 1$, then $\text{E}[\hat{X}|X = 1] = \frac{n}{n+1} < 1$.

Delft University of Technology
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Partial exam EE2S31 SIGNAL PROCESSING
Part 1: May 28th 2021
Block 2: Digital Signal Processing (14:50-15:50)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted. Upload answers during 15:45-16:00.

This block consists of three questions (25 points), more than usual. This will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. No points will be awarded for results without a derivation. Write your name and student number on each sheet.

Question 4 (11 points)

Let us consider the bicycle wheel on Figure 1. We are taking a video of this wheel at a rate of 24 frames per second. Let us assume that we keep the position of the wheel within the video frame steady.

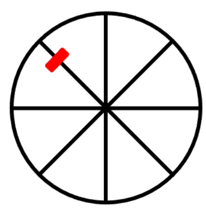


Figure 1

(2 p) (a) What is the maximum angular speed of the wheel that the camera can capture truthfully?

(2 p) (b) At what speed will it appear on the video as if the wheel is standing still? Explain in your own words what is the reason why the wheel appears to stand still!

(2 p) (c) What would happen if we removed the red reflector light? How does your answer to (a) and (b) change?

(3 p) (d) Let's consider now that the wheel is moving at a constant speed and we are capturing the video for infinitely long. The series of values captured by a certain pixel are [1 0 1 0 1 0 ...]. What is the 8-point DFT of this series?

(2 p) (e) In practice, we cannot continue taking the video forever. Let's consider now that we are capturing only 7 frames. Is the 8-point DFT of this series the same as the 8-point DFT of the above infinite series? Why? What about the 8-point DFT of the first 6 frames?

Solution

(a) With 24 frames per second, our sampling rate is $F_s = 24Hz$. The rotation of the wheel gives rise to a periodic signal, with a period equal to the time it takes for the red light to turn around and get back to its original position. At 24Hz the maximum frequency we can capture is $F_{max} = \frac{F_s}{2} = 12$ Hz [1 p]. This means that the wheel can rotate with a maximum of 12Hz, therefore, with a maximum angular speed of $\omega = 2\pi \cdot F_{max} = 2\pi \cdot 12 = 75.4$ rad/s [1 p].

(b) It will appear as if the wheel stands still when the red light arrives back to the same position by the time the next frame is captured, i.e. when the angular speed is the same as the framerate or its integer multiple [1 p], i.e. $K \cdot 24$Hz . Therefore, $f_{still} = K \cdot 2\pi \cdot 24 = K \cdot 150.8$ rad/s [1 p].

(c) If we remove the red light, the wheel is rotationally symmetric: it appears to be the same after only $\frac{1}{8}$ of a full rotation. Therefore, it can move 8 times slower than before to appear the same [1 p]. The maximum speed we can capture is $\frac{F_{max}}{8}$ and it will appear to stand still at $\frac{F_{still}}{8}$ [1 p].

(d) The 8-point DFT is computed, by definition, using:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{kn}{N}},\ k = 0, ..., N-1$$

$$X[0] = \sum_{n=0}^{7} x[n] e^{0} = \sum_{k=0}^{N-1} x[n] = 4 \text{ [1 p]}$$

$$X[1] = \sum_{n=0}^{7} x[n] e^{-j2\pi \frac{n}{8}} = 1 \cdot e^{-j2\pi \frac{0}{8}} + 1 \cdot e^{-j2\pi \frac{2}{8}} + 1 \cdot e^{-j2\pi \frac{4}{8}} 1 \cdot e^{-j2\pi \frac{6}{8}}$$

$$= \sum_{m=0}^{3} (e^{-j2\pi \frac{2}{8}})^m = \frac{1 - (e^{-j2\pi \frac{2}{8}})^4}{1 - e^{-j2\pi \frac{2}{8}}} = \frac{1 - 1}{1 - e^{-j2\pi \frac{2}{8}}} = 0$$

In general [1 p]:

$$X[k] = \sum_{n=0}^{7} x[n] e^{-j2\pi \frac{nk}{8}} = 1 \cdot e^{-j2\pi \frac{0k}{8}} + 1 \cdot e^{-j2\pi \frac{2k}{8}} + 1 \cdot e^{-j2\pi \frac{4k}{8}} 1 \cdot e^{-j2\pi \frac{6k}{8}}$$

If $|e^{-j2\pi \frac{2k}{8}}| \neq 1$, then, similarly as for X[1], the sum of geometric series is 0.

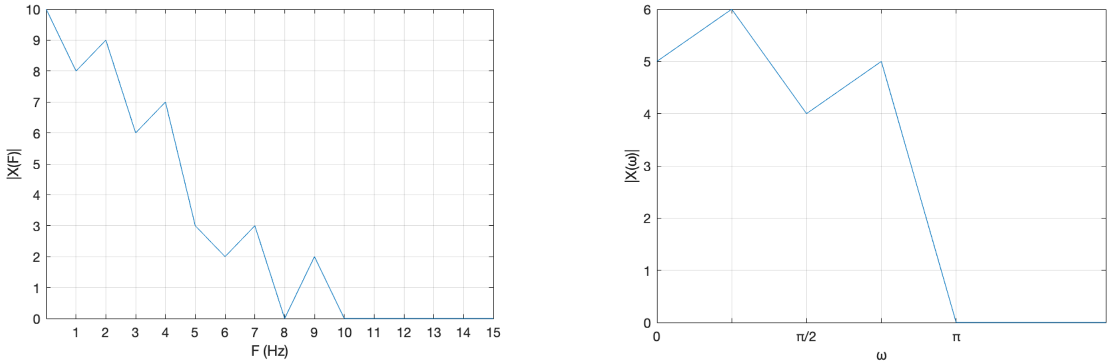
If $e^{-j2\pi \frac{2k}{8}} = 1$, then the above sum is equal to 4. This equation holds for k=0 and k=4.

Therefore, the 8-point DFT is [4 0 0 0 4 0 0 0]. [1 p]

(e) The 6 and 7 long series, respectively, are:
$x_1 = [1010101]$
$x_2 = [101010]$
We have to zero-pad these short sequences in order to take their 8-point DFT. After zero-padding, they become
$x_1^{(z)} = [10101010]$
$x_2^{(z)} = [10101000]$
The periodic extension of $x_1^{(z)}$ is the same as the original infinite series, so its DFT is the same as in (d) [1 p]. This is not true for $x_2^{(z)}$, the corresponding DFT is different [1 p].

Question 5 (7 points)

Given a real analog signal with a spectrum shown in Figure 2a. We want to sample the signal with 20Hz. We know that the signal contains noise above 5Hz.



(a) (b)

Figure 2

(2 p) (a) Design (sketch) the cheapest possible (non-ideal) antialiasing filter with a linear transition band that preserves the noiseless part of signal.

(2 p) (b) Sketch the spectum of the signal after filtering and sampling! Make sure to correctly indicate the magnitude and frequency values as well as the labels of the axes (pay attention to correct sketching in part (a) too)!

(1 p) (c) Let's assume that after further digital processing, the spectrum of our digital signal is as depicted on Figure 2b. Let's represent this spectrum using an 8-point DFT. What are the values of the DFT coefficients $Y[k]$?

(1 p) (d) Let's further filter the signal with a system with frequency response $H[k] = [1\ 0.9\ 0.8\ 0.7\ 0.6\ 0.7\ 0.8\ 0.9]$. What is the DFT of the resulting signal?

(1 p) (e) After filtering using the system $H[k]$ and taking the inverse DFT of the filtered signal, we notice that the first values of the sequence are non-zeros, despite the fact that the original sequence (that corresponds to the spectrum in Figure 2b) are zeros. How do you explain this?

Solution

(a) Figure 3 below depicts the antialiasing filter [2 p].

(b) Figure 4 depicts the spectrum of the signal after filtering and sampling. Notice that frequencies between 5-10Hz are attenuated due to filtering compared to the original spectrum [1 p for periodic spectral image, 1 p for correct frequencies and magnitudes, 1 p for indicating the attenuation between 5-10Hz] .

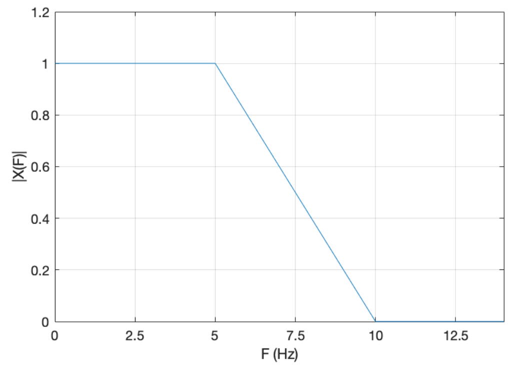


Figure 3

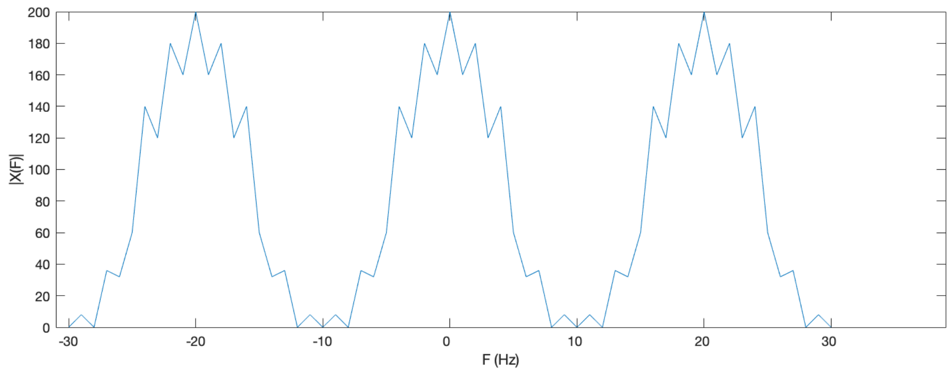


Figure 4

(c) DFT takes equidistant samples from the requency interval $[-\pi \ \pi]$. For real signals, the spectrum is symmetric, and X[k] = X[N-k] for an N-point DFT. X[0] is the sample at 0 frequency. For X[1], ...X[4] we need to take 4 samples between 0 to π (inclusive π). Therefore, based on Figure 2b: X[0] = 5, X[1] = 6, X[2] = 4, X[3] = 5, X[4] = 0, X[5] = X[8-5] = 5, X[6] = 4, X[7] = 6 [1 p].

(d) In frequency domain, we can multiply the DFTs of the signal and the filter to obtain the filtered signal. Therefore, the resulting signal is

$$\begin{bmatrix} 5 \cdot 1 & 6 \cdot 0.9 & 4 \cdot 0.8 & 5 \cdot 0.7 & 0 \cdot 0.6 & 5 \cdot 0.74 \cdot 0.8 & 6 \cdot 0.9 \end{bmatrix}$$

[1 p].

(e) Non-zero values appear due to the circular convolution property: the origin of these values is the last couple of samples of the original sequence [1 p].

Question 6 (7 points)

(1 p) (a) We are sampling a slowly varying signal with $F_s = 1.05Hz$ sampling rate. How long do we need to sample in order to obtain $N = 21$ sample values?

(1 p) (b) We would like to analyse the spectrum of the acquired sequence. What will be the limit of the frequency resolution?

(3 p) (c) We want to compute the DFT of our sequence using FFT. Give 2 alternative FFT-based algorithms that we could use. Outline the major steps of each algorithm!

(2 p) (d) Which option has lower computational complexity? I.e. how many multiplications do we need to perform for each of the two algorithms?

Solution

(a) We need to sample for $T = \frac{n}{F_s} = \frac{21}{1.05Hz} = 20s$ long (i.e. the observation interval) [1 p].

(b) The frequency resolution is limited by the sampling interval and the number of samples, or, in other words, the duration of finite observation interval T. We cannot distinguish 2 frequencies which are closer to each other than $\frac{1}{NT_s} = \frac{1}{T} = \frac{1}{20} = 0.05Hz$ [1 p].

(c) Two alternatives:

- radix-2 FFT [0.5 p]
- Divide and conquer appraoch for $N = LM$ long sequences, here $N = 21$ and $L = 3$, $M = 7$.[0.5 p]

Radix-2 FFT [1 p]:

- zero-pad the sequence till $N_z = 32$
- sort the samples of the sequence in bit-reversed order
- apply radix-2 FFT using 5 stages

Divide and conquer appraoch for $N = LM$ long sequences [1 p]:

- organize the samples into a 3-by-7 matrix, filling the matrix, for example, column-wise
- compute 3-point DFTs of each row, resulting in a 3-by-7 matrix again
- multiply each element of the matrix with an appropriate phase factor
- compute 7-point DFTs of each column
- read out the DFT coefficients row-wise.

(d) For the radix-2 algorihtm, each 2-point DFT (butterfly) requires one (complex) multiplication. Each stage requires 32/2=16 butterflies and we have 5 stages. Therefore, we need 80 (complex) multipliations [1 p].

For the other algorithm: each N-point DFT take N^2 (complex) multiplications. First, we need 7 3-point DFTs. Then, we need N multiplications with the phase factors. Finally, we need 3 7-point DFTs. That is, $7 \cdot 3^2 + 21 + 3 * 7^2 = 231$. [1 p].