Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

exam EE2S31 SIGNAL PROCESSING
Resit: 29 July 2019 (13:30–16:30)

Closed book; two sides A4 of handwritten notes permitted

This exam consists of 5 questions (36 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.

Question 1 (9 points)

The random variables X and Y have the joint probability density function (pdf)

$$f_{X,Y}(x,y) = \begin{cases} c\,, & \text{for} \quad 0 \le x \le y^2 \le 1 \\ 0\,, & \text{otherwise.} \end{cases}$$

(2 p) (a) Make a sketch of the area where the pdf is non-zero and calculate the constant c.

(2 p) (b) Calculate the marginal pdfs $f_X(x)$ and $f_Y(y)$ as a function of parameter c.

(3 p) (c) Calculate $E[X|X \ge \frac{1}{4}]$.

(2 p) (d) Calculate $E[X|Y]$.

Answer

(2 p) (a)

$$\int_y \int_x f_{X,Y}(x,y)dxdy = \int_0^1 \int_0^{y^2} cdxdy = \int_0^1 [cx]_0^{y^2}\, dy = \int_0^1 cy^2 dy = \left[\frac{c}{3}y^3\right]_0^1 = 1$$

or:

$$\int_x \int_y f_{X,Y}(x,y)dydx = \int_0^1 \int_{\sqrt{x}}^1 cdydx = \int_0^1 [cy]_{\sqrt{x}}^1\, cdx = c\int_0^1 (1-\sqrt{x})dy = c\left[x - \frac{2}{3}x^{3/2}\right]_0^1 = 1$$

From both expressions it follows that $c = 3$.

(2 p) (b) $f_X(x) = \int_y f_{X,Y}(x,y)dy = \int_{x^{1/2}}^1 cdy = [cy]_{x^{1/2}}^1 = c(1-\sqrt{x})$ for $0 \le x \le 1$.

$f_Y(y) = \int_x f_{X,Y}(x,y)dx = \int_0^{y^2} cdy = [cx]_0^{y^2} = cy^2$ for $0 \le y \le 1$.

(3 p) (c) $P(X \ge 1/4) = \int_{x=\frac{1}{4}}^{x=1} \int_{y=\sqrt{x}}^1 cdydx = \int_{x=\frac{1}{4}}^{x=1} [cy]_{y=\sqrt{x}}^1 dx = \int_{x=1/4}^{x=1} [c - c\sqrt{x}]dx = [cx - \frac{2}{3}cx^{3/2}]_{1/4}^1 = c(\frac{1}{3} - (\frac{1}{4} - \frac{1}{12})) = \frac{c}{6}$

$$f_{X,X\ge 1/4}(x) = \begin{cases} \frac{c(1-\sqrt{x})}{\frac{c}{6}} = 6(1-\sqrt{x})\,, & \text{for} \quad 1/4 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

$E[X|X \ge 1/4] = \int_{1/4}^1 x6(1-\sqrt{x})dx = 6\left[\frac{1}{2}x^2 - \frac{2}{5}x^{5/2}\right]_{1/4}^1 = \frac{6}{10} - \frac{9}{80} = \frac{39}{80}$.

(2 p) (d) $E[X|Y] = \int_0^{y^2} x\frac{1}{y^2}dx = \left[\frac{x^2}{2}\frac{1}{y^2}\right]_0^{y^2} = \frac{y^2}{2}$

Question 2 (10 points)

Given is the stochastic process $X(t) = A + t$ with random variable A uniformly distributed in the (continuous) interval $[0, 2]$.

(1 p) (a) Sketch three different realizations of process $X(t)$.

(3 p) (b) Calculate the cumulative distribution function (CDF) $F_{X(t)}(x)$, as well as the probability density function (PDF) $f_{X(t)}(x)$.

(2 p) (c) Calculate the expected value $E[X(t)]$, the autocorrelation function $R(t, \tau)$, and argue whether or not this process is wide sense stationary (WSS).

Process $X(t)$ is used as an input to a system with impulse response $h(t)$, having an output $Y(t)$.

(1 p) (d) Under which conditions is the output of such a system WSS?

(1 p) (e) Under which conditions are the input and output of such a system jointly WSS?

The impulse response $h(t)$ is given by

$$h(t) = \begin{cases} 1 & \text{for} \quad 0 \leq t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(2 p) (f) Calculate $E[Y(t)]$, i.e., the expected value of the output $Y(t)$ when the input is given by $X(t) = A + t$.

Answer

(1 p) (a)

(3 p) (b)

$$\begin{aligned} F_{X(t)}(x) &= P(X(t) \leq x) \\ &= P(A + t \leq x) \\ &= P(A \leq x - t) \\ &= \int_0^{x-t} \frac{1}{2} da \\ &= \left[\frac{a}{2}\right]_0^{x-t} \\ &= \frac{x - t}{2}, \end{aligned}$$

for $t \leq x \leq 2 + t$.

Altogether,

$$F_{X(t)}(x) = \begin{cases} 1\,, & \text{for } x \geq 2 + t \\ \frac{x-t}{2}\,, & \text{for} \quad t \leq x \leq 2 + t \\ 0\,, & \text{otherwise.} \end{cases}$$

$$f_{X(t)}(x) = \frac{dF_{X(t)}(x)}{dx} = \begin{cases} \frac{1}{2}\,, & \text{for} \quad t \leq x \leq 2 + t \\ 0\,, & \text{otherwise.} \end{cases}$$

(2 p) (c) $E[X(t)] = E[A] + t = 1 + t$ and $R(t, t+\tau) = E[(A+t)(A+t+\tau)] = t^2 + t\tau + E[A^2] + E[A](2t+\tau) = t^2 + t\tau + \frac{4}{3} + (2t+\tau)$.

The process is not WSS as both the expected value and the autocorrelation function depend on the actual time t (they are not shift invariant).

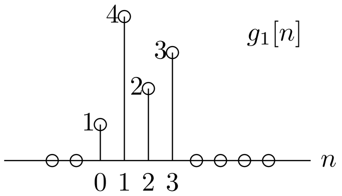
(1 p) (d) If the input process is WSS and the system is LTI.

(1 p) (e) If the input and output are WSS, and, the cross correlation R_{XY} depends only on the time difference τ and not on the actual time t.

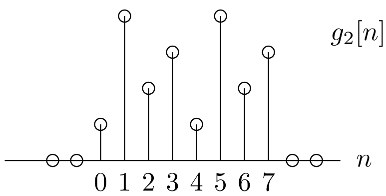
(2 p) (f) $E[Y(t)] = \int_{-\infty}^{\infty} h(u) E[X(t-u)]\, du = \int_0^1 (E[A] + (t-u)) du = 1 + t - 1/2 = t + 1/2$.

Question 3 (5 points)

Let $G_1(\omega)$ be the DTFT of the sequence $g_1[n]$ in figure (a):



(a)



(b)

(a) Determine the DTFT of the other sequence, $g_2[n]$, in terms of $G_1(\omega)$.

(Note: do not evaluate $G_1(\omega)$ explicitely!)

(b) Take $N = 8$. What is the DFT of $g_1[n]$ $(n = 0, \cdots, N-1)$, expressed in terms of $G_1(\omega)$?

(c) How does the DFT of $g_1[n]$ change if we take $N = 16$ or $N = 4$? Explain using a drawing of the spectrum.

(d) What changes if the DFT of $g_2[n]$ is taken (using $N = 8$)? Explain using a drawing.

Answer

(1 p) (a)

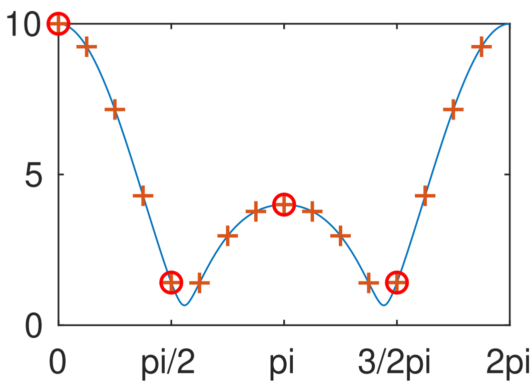
$$g_2[n] = g_1[n] + g_1[n-4] \quad \Rightarrow \quad G_2(\omega) = G_1(\omega) + e^{-4j\omega} G_1(\omega)$$

(1 p) (b)

$$G_1^{dft}[k] = G_1(\frac{2\pi}{8}k), \qquad k = 0, \cdots, 7.$$

These are samples of $G_1(\omega)$ for multiples of $\omega = 2\pi/N$. This is valid because $g_1[n]$ is zero outside the DFT interval.

(1.5 p) (c) Fewer, resp. more samples located on the same curve $G_1(\omega)$ (zero-padding effect).

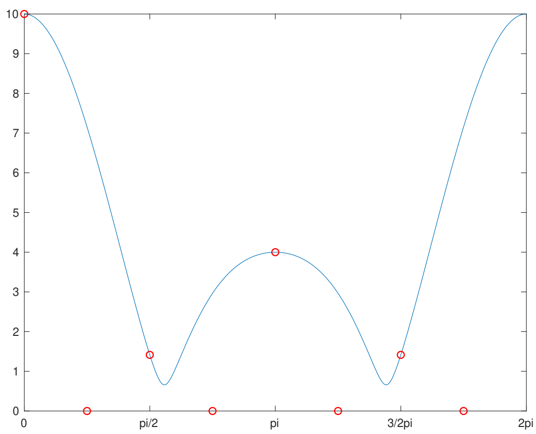


(1.5 p) (d)

$$G_2^{dft}[k] = G_1(\frac{2\pi}{8}k) + e^{-4j\frac{2\pi}{8}k} G_1(\frac{2\pi}{8}k) = G_1^{dft}[k] + (-1)^k G_1^{dft}[k]$$

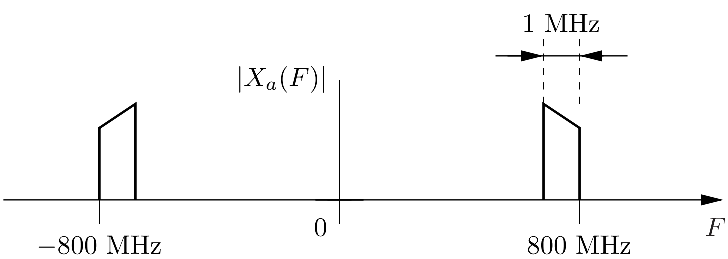
For k even, this is equal to twice $G_1^{dft}[k]$ for $N = 8$ samples, hence equal to the spectrum for $N = 4$ samples. For k odd, this is equal to zero.

Apart from the factor 2, we obtain the same spectrum as for the DFT of $g_1[n]$ with $N = 4$, and with intermittent samples 0.



Question 4 (6 points)

We are given a real analog signal $x_a(t)$ with spectrum given as follows:

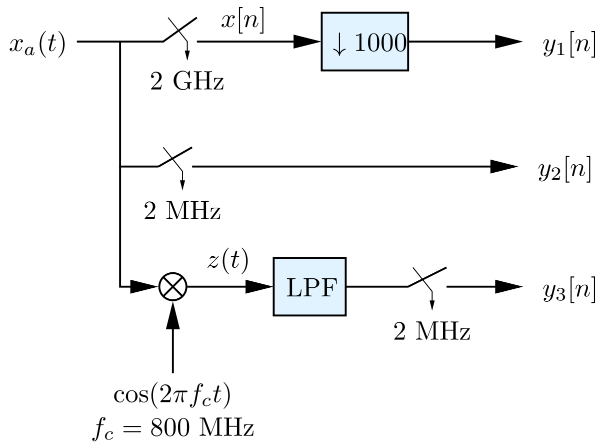


Think e.g. of a GSM signal at a carrier frequency slightly below 800 MHz.

(a) For this case, what is the Nyquist rate?

(b) What sample rate do you need at least to describe the signal?

We consider three techniques to demodulate and sample the signal.



(c) For signal $y_1[n]$, we sample the signal $x_a(t)$ at a rate of 2 GHz, and subsequently downsample by a factor 1000.

Draw the resulting spectum $Y_1(F)$. On the frequency axis, specify both ω and the corresponding 'real' frequencies F.

(d) For signal $y_2[n]$, we sample $x_a(t)$ at a rate of 2 MHz.

Draw the resulting spectrum $Y_2(F)$.

(e) For signal $y_3[n]$, we first apply a demodulation with $f_c = 800$ MHz, i.e., we multiply $x_a(t)$ by $\cos(2\pi f_c t)$, next we apply a suitable low pass filter (LPF), and sample at 2 MHz.

- Draw the spectrum $Z(F)$. (*Hint:* split cos into the sum of two complex exponentials.)
- What are suitable parameters for the LPF to prevent aliasing?
- Draw the resulting spectrum $Y_3(F)$.

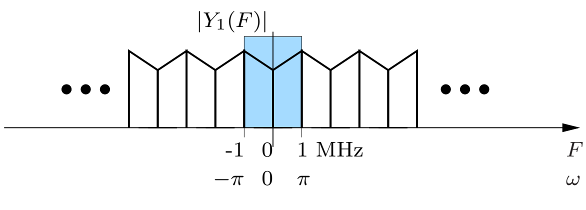
Answer

(0.5 p) (a) 1600 MHz

(0.5 p) (b) 2 MHz (Alternatively: 1 MHz, but using complex samples.)

(1 p) (c) Note that $y_1[n]$ is identical to $y_2[n]$. We sampled faster, but then throw away most samples.

It is probably easier to directly determine the spectrum of $y_2[n]$, as follows.



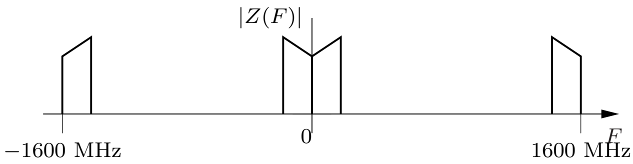
(1 p) (d) Identical to answer (c).

The signal between 799 and 800 MHz returns at multiples of 2 MHz, i.e. also at the interval -1 until 0 MHz. The signal between -800 and -799 MHz returns at multiples of 2 MHz, i.e. also at the interval 0 until 1 MHz.

(3 p) (e) $\cos(2\pi f_c t) = \frac{1}{2}(e^{j2\pi f_c t} + e^{-j2\pi f_c t})$.

The first term shifts the spectrum to the right by f_c; the component at -800 MHz shifts to 0, and the component at 800 MHz shifts to 1600 MHz.

he second term shifts the spectrum to the left by f_c; the component at 800 MHz shifts to 0, and the component at -800 MHz shifts to -1600 MHz.

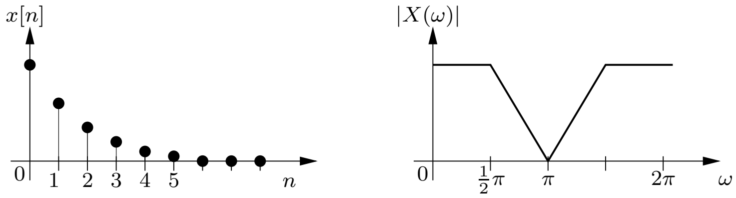


The lowpass filter should retain only the component around 0 MHz. Specs: passband from 0 to 1 MHz, stop band from 1599 MHz.

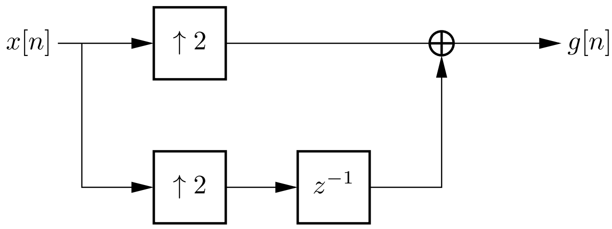
After that, sampling at 2 MHz results in the same spectrum as we had for $y_1[n]$ and $y_2[n]$.

Question 5 (6 points)

We are given a signal $x[n]$ with the following amplitude spectrum:



We consider the following system:



(a) Make a plot of the series $g[n]$.

(b) How is the spectrum $G(\omega)$ of $g[n]$ related to that of $x[n]$? (give a formula.)

Also make a drawing of the amplitude spectrum $|G(\omega)|$.

A "classical" movie projector in a cinema displays 24 images per second. Every image is projected using a short flash of light. This results in a flickering effect. To reduce this, every image is displayed twice (48 images per second).

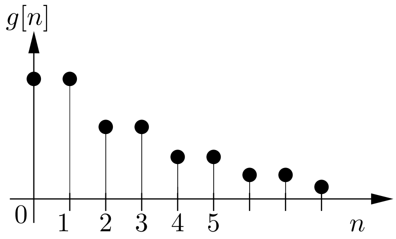
To model the movie, we consider an arbitrary pixel and represent it by a time series $x[n]$. To model the observation by the audience, we model the eye as an ideal D/A converter, followed by a lowpass filter with a passband until 20 Hz and a stop band from 35 Hz.

(c) Give a block scheme representing the "double" projection process of $x[n]$ until the observation.

(d) Motivate why the double projection results in less flicker. (Use frequency spectra.)

Answer

(1 p) (a)

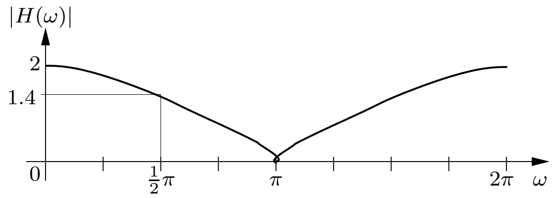


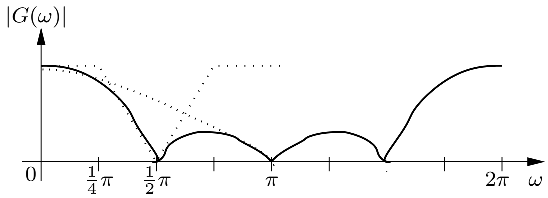
(2 p) (b) Denote by $Y(z)$ the signal after upsampling by a factor 2. Then

$$G(z) = Y(z)(1 + z^{-1})$$

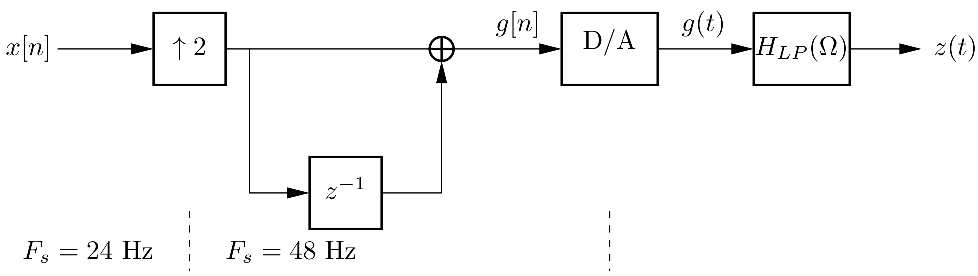
$$G(\omega) = Y(\omega)(1 + e^{-j\omega}) = X(2\omega)(1 + e^{-j\omega})$$

The filter $H(z) = 1 + z^{-1}$ has a zero at $z = -1$ (corresponding to $\omega = \pi$), it is a weak lowpass filter. The power spectrum of $H(z)$ is $|H(\omega)|^2 = (1+e^{-j\omega})(1+e^{j\omega}) = 2+2\cos(\omega)$. The spectrum $G(\omega)$ is the multiplication of the spectrum $Y(\omega)$ with that of this filter. This suppresses the frequencies around $\omega = \pi$.

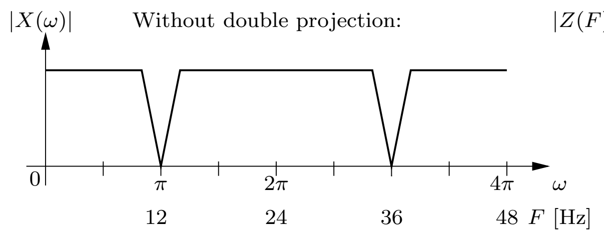


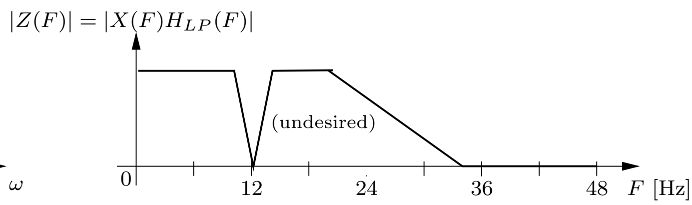


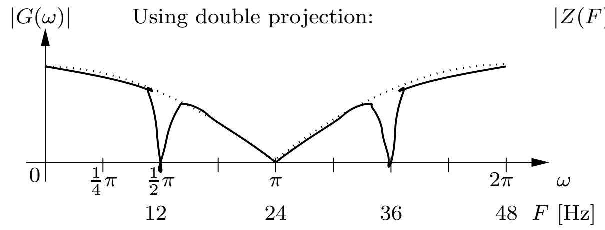
(1 p) (c)

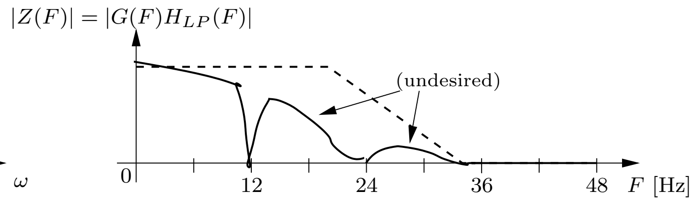


(2 p) (d)









Flicker corresponds to aliasing; the eye is not a very good anti-aliasing filter. Ideally all frequencies above 12 Hz should be suppressed.

By double projection, in the frequency spectrum of $G(\Omega)H_{LF}(\Omega)$ the frequencies between 12 and 36 Hz will be suppressed quite nicely by the filter $1 + z^{-1}$, and the frequencies above 36 Hz by the "eye" filter $H_{LF}(F)$. As a result, the observer perceives an analog signal with spectrum mostly until 12 Hz, the original frequency content of the movie.

Without this process, we clearly have more frequency content between 12 and 36 Hz, the flicker. These high frequencies do not contain useful information (aliasing).

(For the same reasons, also TVs and monitors are sold boasting higher update rates, e.g. 60 or 70 Hz.)