Mid-term exam EE2S31 Signaalbewerking - Answers

May 30th, 2017

Answer each question on a **separate sheet**. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet. It is allowed to answer in Dutch or English.

Question 1 (10 points)

(2 p) (a) $\int_0^{3/2} \int_{2y}^3 cdxdy = \int_0^{3/2} [cX]_{2y}^3 dy = \int_0^{3/2} c(3-2y)dy = c[3y-y^2]_0^{3/2} = c(9/2-9/4) = 9c/4 = 1$. Based on this we can calculate that $c = 4/9$

(2 p) (b)

$$f_X(x) = \int_0^{x/2} cdy = \frac{1}{2}cx$$

$$f_Y(y) = \int_{2y}^3 cdx = 3c - 2cy.$$

X and Y are not independent as $f_X(x)f_Y(y) \neq f_{X,Y}(x,y)$.

(2 p) (c) The blind estimator is given by $\hat{X}_1 = E[X] = \int_0^3 \frac{cx^2}{2}dx = \frac{cx^3}{6} = \frac{27c}{6} = \frac{9c}{2}$.

(2 p) (d) $P(X > 2) = \int_2^3 \frac{1}{2}cxdx = [\frac{1}{4}cx^2]_2^3 = \frac{5c}{4}$

$$f_{X,X>2}(x) = \begin{cases} f_X(x)/P(X>2) & \text{for } x > 2 \\ 0 & \text{otherwise.} \end{cases} = \begin{cases} \frac{4}{10}x & \text{for } x > 2 \\ 0 & \text{otherwise.} \end{cases}$$

$E[X|X > 2] = \int_2^3 \frac{4x^2}{10}dx = [4x^3/30]_2^3 = \frac{38}{15}$

In addition to the knowledge on the distributions, we also observe the realisation of random variable Y

(2 p) (e)

$$f_{X|Y}(x|y) = \begin{cases} f_{X,Y}(x,y)/f_Y(y) & \text{for } 2y \le x \le 3 \\ 0 & \text{otherwise.} \end{cases} = \begin{cases} \frac{1}{3-2y} & \text{for } 2y \le x \le 3 \\ 0 & \text{otherwise.} \end{cases}$$

Estimator $\hat{X}_3$ is given by $E[X|Y] = \int_{2y}^3 \frac{x}{3-2y}dx = [\frac{x^2}{6-4y}]_{2y}^3 = \frac{9}{6-4y} - \frac{4y^2}{6-4y}$

Question 2 (16 points)

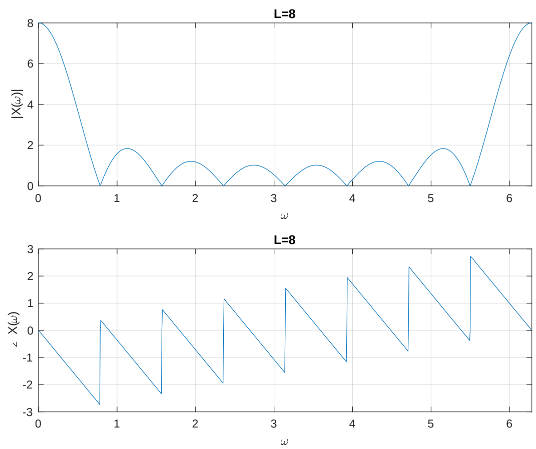
(3 p) (a)

$$X(\omega) = \sum_{n=0}^{L-1} e^{-j\omega n} = \frac{1 - e^{-j\omega L}}{1 - e^{-j\omega}} = e^{-j\omega(L-1)/2} \frac{\sin(\omega L/2)}{\sin(\omega/2)}.$$

The function is continuous in ω (x is non-periodic) and periodic (x is discrete-time).

(2 p) (b)

$$|X(\omega)| = \left| \frac{\sin(\omega L/2)}{\sin(\omega/2)} \right|, \quad \angle X(\omega) = \angle \frac{\sin(\omega L/2)}{\sin(\omega/2)} - \omega(L-1)/2.$$



(3 p) (c)

$$Y(k) = \sum_{n=0}^{L-1} e^{-j\frac{2\pi}{N}kn} = \frac{1 - e^{-j\frac{2\pi}{N}kL}}{1 - e^{-j\frac{2\pi}{N}k}} = e^{-j\frac{\pi}{N}k(L-1)} \frac{\sin(\frac{\pi}{N}kL)}{\sin(\frac{\pi}{N}k)}.$$

The function is discrete in frequency (line spectrum since y is of finite length) and periodic (y is discrete-time).

(2 p) (d) $Y(k)$ are samples of $X(\omega)$, taken in the interval $[0, 2\pi)$ with inter-sample distance $\frac{2\pi}{N}$. That is,

$$Y(k) = X\left(\frac{2\pi}{N}k\right), \quad k = 0, \ldots, N-1.$$

(2 p) (e) If $N \geq L$ there is no time-domain aliasing and perfect reconstruction is possible. We have

$$X(\omega) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k) G\left(\omega - \frac{2\pi}{N}k\right),$$

where

$$G(\omega) = e^{-j\omega\frac{N-1}{2}} \frac{\sin(\frac{\omega N}{2})}{\sin(\frac{\omega}{2})}.$$

(2 p) (f) If $N = L$, we have

$$Y(k) = e^{-j\frac{\pi}{N}k(L-1)} \frac{\sin(\frac{\pi}{N}kL)}{\sin(\frac{\pi}{N}k)} = (-1)^k e^{j\frac{\pi}{L}k} \frac{\sin(\pi k)}{\sin(\frac{\pi}{L}k)}, \quad k = 0, \ldots, N-1,$$

so that $Y(k) = L\delta(\mathrm{k})$.

(2 p) (g) The value of N determines the number of samples taken in the frequency domain. If $N = L$, we only taken L samples and by inspection of the figure above, we see that $Y(k) = L\delta(k)$, since we sample $X(\omega)$ exactly at the zero-crossings, except for $k = 0$, which gives us the value $Y(0) = X(0) = L$. Increasing N, we get more and more samples of the spectrum, each separated by $\frac{2\pi}{N}$.

Question 3 (8 points)

(2 p) (a) 1)

$$P(Z \geq 3) \leq \frac{E[Z]}{3} = \frac{5/6}{3} = \frac{5}{18}$$

2)

$$P(Z \geq 3) = P(Z - E[Z] \geq 3 - E[Z]) \leq P(|Z - E[Z]| \geq 3 - E[Z]) = P(|Z - E[Z]| \geq 3 - \frac{5}{6}) =$$

$$P(|Z - E[Z]| \geq \frac{13}{6}) \leq Var[Z]/(\frac{13}{6})^2 = \frac{1/4 + 1/9 + 2 * cov(X, Y)}{(\frac{13}{6})^2} = 1/13$$

The Chebyshev bound is more strict as it uses not only the first moment, but also the second moment to bound the probability.

(2 p) (b) $\phi_{X_i}(s) = \int_0^\infty e^{sx} \lambda_i e^{-\lambda_i x_i} dx_i = \frac{\lambda_i}{\lambda_i - s}$.

(2 p) (c) $\phi_Z(s) = \frac{\lambda_1}{\lambda_1 - s} \frac{\lambda_2}{\lambda_2 - s}$ No apply fraction expansion:
$\frac{\lambda_1}{\lambda_1 - s} \frac{\lambda_2}{\lambda_2 - s} = \frac{A}{\lambda_1 - s} + \frac{B}{\lambda_2 - s}$ with
$A = \frac{(\lambda_1 - s)\lambda_1}{\lambda_1 - s} \frac{\lambda_2}{\lambda_2 - s}|_{s = \lambda_1} = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}$ and
$B = \frac{(\lambda_2 - s)\lambda_1}{\lambda_1 - s} \frac{\lambda_2}{\lambda_2 - s}|_{s = \lambda_2} = -\frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}$

$$\phi_Z(s) = \frac{\lambda_2}{\lambda_2 - \lambda_1} \frac{\lambda_1}{\lambda_1 - s} - \frac{\lambda_1}{\lambda_2 - \lambda_1} \frac{\lambda_2}{\lambda_2 - s}.$$

Hint: You might want to make use of fraction expansion.

(2 p) (d) Based on the result from the previous question we can apply the Laplace transform to go back from the MGF to the pdf $f_Z(z)$
$f_Z(z) = \frac{\lambda_2 \lambda_1 e^{-\lambda_1 z}}{\lambda_2 - \lambda_1} - \frac{\lambda_1 \lambda_2 e^{-\lambda_2 z}}{\lambda_2 - \lambda_1} = \frac{\lambda_2 \lambda_1}{\lambda_2 - \lambda_1} \left(e^{-\lambda_1 z} - e^{-\lambda_2 z} \right)$

$$P(Z > 3) = \int_3^\infty f_Z(z) dz = \int_3^\infty \frac{\lambda_2 \lambda_1}{\lambda_2 - \lambda_1} \left(e^{-\lambda_1 z} - e^{-\lambda_2 z} \right) dz =$$

$$= \frac{\lambda_2 \lambda_1}{\lambda_2 - \lambda_1} \left(\frac{1}{\lambda_1} e^{-\lambda_1 3} - \frac{1}{\lambda_2} e^{-\lambda_2 z} \right)$$