Mid-term exam EE2S31 Signaalbewerking

May 30th, 2017

Answer each question on a **separate sheet**. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet. It is allowed to answer in Dutch or English.

Question 1 (10 points)

Given is the joint probability density function of two random variables X and Y:

$$f_{X,Y}(x,y) = \begin{cases} c & \text{for} \quad 0 \leq y \leq \frac{1}{2}x \quad \text{and } x \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

In this question we study the use of different estimators for random variable X.

(2 p) (a) Calculate the value of constant c.

(2 p) (b) Show that $f_X(x)$ and $f_Y(y)$ are given by

$$f_X(x) = \frac{1}{2}cx,$$

and

$$f_Y(y) = 3c - 2cy,$$

and argue whether the random variables X and Y are independent or not.

The blind estimator $\hat{X}_1$ for random variable X estimates X based on the marginal pdf $f_X(x)$ only, and minimizes the mean-squared error (MSE) $E[(X - \hat{X}_1)^2]$.

(2 p) (c) Determine the blind estimator $\hat{X}_1$.

We have prior knowledge that $x \in A$ with $A = \{X > 2\}$.

(2 p) (d) Determine the estimator $\hat{X}_2$ that takes this prior information into account by minimizing the MSE $E\left[(X - \hat{X}_2)^2 | X > 2\right]$.

In addition to the knowledge on the distributions, we also observe the realisation of random variable Y.

(2 p) (e) Calculate the estimator $\hat{X}_3$ that minimizes the MSE $E\left[(X - \hat{X}_3)^2 | Y = y\right]$.

Question 2 (16 points)

Consider the (infinitely long) discrete-time signal

$$x(n) = \begin{cases} 1, & n = 0, \ldots, L-1, \\ 0, & \text{otherwise.} \end{cases}$$

(3 p) (a) Show that the Fourier transform $X(\omega)$ of x is given by

$$X(\omega) = e^{-j\omega(L-1)/2} \frac{\sin(\omega L/2)}{\sin(\omega/2)}.$$

Is this function continuous or not? And periodic or not? Motivate your answer.

(2 p) (b) Give an expression for the magnitude and phase of $X(\omega)$ and sketch the result for $\omega \in [0, 2\pi)$.

Consider the finite-length signal y defined as

$$y(n) = \begin{cases} 1, & n = 0, \ldots, L-1, \\ 0, & n = L, \ldots, N-1, \end{cases}$$

with $N \geq L$.

(3 p) (c) Show that the discrete Fourier transform $Y(k)$ of y is given by

$$Y(k) = e^{-jk\pi \frac{L-1}{N}} \frac{\sin(k\pi L/N)}{\sin(k\pi/N)}.$$

Is this function continuous or not? And periodic or not? Motivate your answer.

(2 p) (d) What is the relation between $X(\omega)$ and $Y(k)$? Motivate your answer.

(2 p) (e) For what values of N can we recover x from the samples $Y(k)$ perfectly and give a reconstruction formula for $X(\omega)$.

(2 p) (f) Assume we have $N = L$. Show that $Y(k) = L\delta(k)$.

(2 p) (g) The effect of increasing N is that we add additional zeros to the nonzero part of y (zero padding). What is the effect of zero padding in the frequency domain?

Question 3 (8 points)

Given are the non-negative random variables X_1 and X_2. The expected values and variances for X_1 and X_2 are given by

$$E[X_1] = \frac{1}{\lambda_1} = \frac{1}{2} \text{ and } Var[X_1] = \frac{1}{\lambda_1^2} = \frac{1}{4},$$

and

$$E[X_2] = \frac{1}{\lambda_2} = \frac{1}{3} \text{ and } Var[X_2] = \frac{1}{\lambda_2^2} = \frac{1}{9},$$

respectively. In addition we know that X_1 and X_2 are uncorrelated, that is, $cov(X_1, X_2) = 0$.
The random variable Z is defined as

$$Z = X_1 + X_2.$$

In this question we are interested in calculating the probability $P(Z > 3)$.

(2 p) (a) Give two different bounds on the probability $P(Z > 3)$: 1) using the Markov inequality and 2) using the Chebyshev inequality. Indicate which of these two bounds is most strict and also explain why this is the case.

The random variables X_1 and X_2 are both exponentially distributed and independent from each other. The probability density function (pdf) of an exponentially distributed random variable X_i is given by

$$f_{X_i}(x_i) = \begin{cases} \lambda_i e^{-\lambda_i x_i} & \text{for} \quad x_i \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(2 p) (b) Calculate the moment generating function $\phi_{X_i}(s)$ and show that it equals $\phi_{X_i}(s) = \frac{\lambda_i}{\lambda_i - s}$.

(2 p) (c) Calculate the moment generating function $\phi_Z(s)$ of $Z = X_1 + X_2$, and show that $\phi_Z(s)$ can be written as

$$\phi_Z(s) = \frac{\lambda_2}{\lambda_2 - \lambda_1} \frac{\lambda_1}{\lambda_1 - s} + \frac{\lambda_1}{\lambda_1 - \lambda_2} \frac{\lambda_2}{\lambda_2 - s}.$$

Hint: You might want to make use of fraction expansion.

(2 p) (d) Calculate the probability $P(Z > 3)$
Hint: First calculate the pdf $f_Z(z)$ using the expression for $\phi_Z(s)$ given above.