Exam EE2S31 Signaalbewerking

June 24th, 2016

Answer each question on a **separate sheet**. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet. It is allowed to answer in Dutch or English.

Question 1 (8 points)

Given are two exponentially distributed random variables X and N with probability density functions (pdfs)

$$f_X(x) = \lambda e^{-\lambda x} \quad \text{for } x \geq 0$$

and

$$f_N(x) = \lambda e^{-\lambda n} \quad \text{for } n \geq 0.$$

(2 p) (a) Calculate the moment generating function $\Phi_X(s) = E[e^{sX}]$.

(2 p) (b) Determine $E[X]$ and $E[X^2]$.

Although we are interested in X, we can only observe the process

$$Y = X + N.$$

Therefore, we would like to derive an estimator for X based on the observed realization of process Y.

(1 p) (c) Show that the process Y has an Erlang distribution, that is, show that its pdf equals

$$f_Y(y) = \begin{cases} \lambda^2 y e^{-\lambda y} & \text{for} \quad y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Hint: You might want to make use of the fact that the moment generating function of an Erlang distribution $f_Y(y)$ is given by $\Phi_Y(s) = \left(\frac{\lambda}{\lambda - s}\right)^2$

The joint pdf $f_{X,Y}(x, y)$ between processes X and Y is given by

$$f_{X,Y}(x, y) = \lambda^2 e^{-\lambda y} \quad \text{for} \quad 0 \leq x \leq y \leq \infty.$$

(3 p) (d) Determine the MMSE estimator $E[X|y]$.

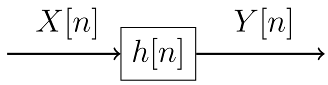

Figure 1: Given LTI system.

Question 2 (8 points)

Consider an application where a signal $x[n]$ is communicated from a transmitting device to a receiving device. Signal $x[n]$ is considered to be a realization of the stochastic zero-mean process $X[n]$ with autocorrelation function

$$R_X[k] = \sigma_X^2\delta[k+1] + 3\sigma_X^2\delta[k] + \sigma_X^2\delta[k-1].$$

For efficiency, prior to transmission, process $X[n]$ is first decorrelated with a filter with impulse response $h[n]$, leading to a process $Y[n]$. This is visualised by the LTI system in Figure 1.

(2 p) (a) Give the magnitude response $|H(f)|$ of the filter that leads to a decorrelated process $Y[n]$ with variance σ_X^2.

Assume now that the impulse reponse $h[n]$ is given by $h[n] = \delta[k-1]$, while $R_Y[k]$ is now unknown and $R_X[k]$ is still given by

$$R_X[k] = \sigma_X^2\delta[k+1] + 3\sigma_X^2\delta[k] + \sigma_X^2\delta[k-1].$$

(3 p) (b) Give the crosscorrelation function $R_{XY}[k]$ and the autocorrelation function $R_Y[k]$ for this situation.

Now we consider an AR-system with $X[n]$ an uncorrelated zero-mean Gaussian noise process with variance σ_X^2 and IIR impuls response $h[n] = a^n u[n]$, with $u[n]$ the unit step function and $|a| < 1$.

(1 p) (c) Calculate the crosscorrelation function $R_{XY}[k]$.

(2 p) (d) Calculate the autocorrelation function $R_Y[k]$. *Hint: You might want to use the generalized expression for the geometric series, given by*

$$\sum_{k=a}^{b} r^k = \frac{r^a - r^{b+1}}{1-r}.$$

for $|r| < 1$.

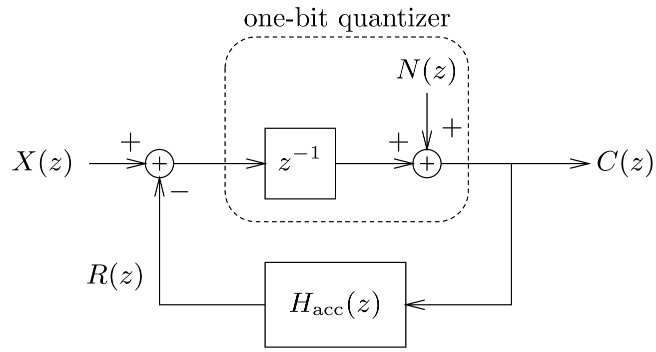

Figure 2: Block diagram of a one-bit delta modulator.

Question 3 (9 points)

Consider the oversampled one-bit D/A convertor of which the discrete-time computational model is depicted in Figure 2. The A/D convertor is referred to as a *delta modulator* (DM). In this figure the combination of the unit delay and the noise signal $N(z)$, which represents quantisation noise, models the process of sampling and one-bit quantisation. The causal LTI filter $H_{\text{acc}}(z)$ is an accumulator, the digital equivalent of an integrator, of which the transfer function is given by

$$H_{\text{acc}}(z) = \frac{z}{z-1}.$$

The output $C(z)$ of the DM is a one-bit code (± 1). The reference signal $R(z)$ is a staircase approximation of the input signal $X(z)$. As a consequence, the decoder of the DM consists of an accumulator, followed by low-pass filtering to reject the out-of-band components due to oversampling.

(1 p) (a) What would be the advantage of quantising the signal as depicted in Figure 2 as compared to directly quantising $X(z)$?

(1 p) (b) Explain why oversampling improves the SNR. Motivate why every doubling of the sampling frequency results in approximately 3 dB SNR improvement.

The total transfer function of the DM can be expressed as

$$R(z) = H_X(z)X(z) + H_N(z)N(z),$$

where $H_X(z)$ and $H_N(z)$ are the transfer functions of the input signal $X(z)$ and quantisation-error signal $N(z)$ to the reference (approximation) signal $R(z)$, respectively.

(2 p) (c) Give expressions for the transfer functions $H_X(z)$ and $H_N(z)$.

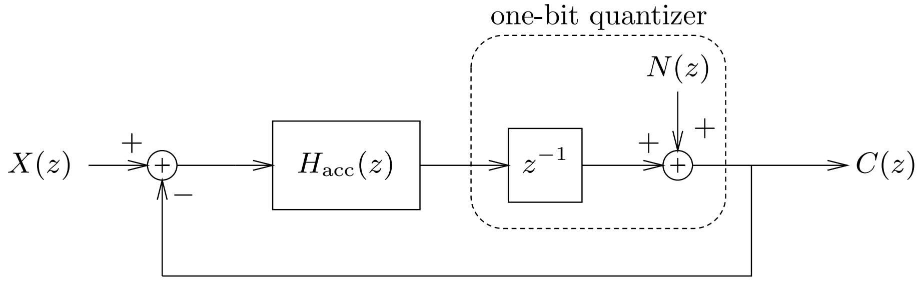

Figure 3: Block diagram of a one-bit sigma-delta modulator.

(1 p) (d) Give expressions for and a sketch of the magnitude response $|H_X(\omega)|$ and $|H_N(\omega)|$.

We can further improve the signal-to-noise ratio considerably by implementing noise shaping. To reach this, one "colors" the noise such that most of the noise power is shifted outside the base band (the band of interest), so that it can be filtered afterwards. Noise shaping can be implemented by using the *sigma-delta modulator* (SDM), of which the discrete-time computational model is depicted in Figure 3. The decoder in this case is simply a low-pass filter to reject the out-of-band components.

(2 p) (e) Give expressions for the transfer functions $H_X(z)$ and $H_N(z)$ for the SDM, the transfer functions of $X(z)$ and $N(z)$ to the output $C(z)$, respectively.

(1 p) (f) Give expressions for and a sketch of the magnitude response $|H_X(\omega)|$ and $|H_N(\omega)|$ for the SDM.

With the SDM, each doubling of the sample frequency results in an approximate gain of the SNR of 9 dB (3 from oversampling and 6 from noise shaping).

(1 p) (g) If we wish to obtain an output SNR of 96 dB (SNR for standard 16-bit PCM) with the 1-bit SDM, what should be the minimal oversampling rate to achieve this?

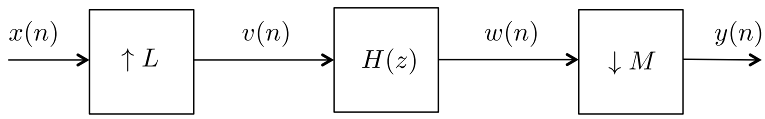

Figure 4: Block diagram of an L/M sample-rate convertor.

Question 4 (7 points)

Consider the sample-rate conversion of an audio signal of which the block diagram is depicted in Figure 4. The original audio signal $x(n)$ is sampled at a sampling frequency of 48 kHz and we would like to convert this to a sampling frequency of 32 kHz.

(1 p) (a) Explain in words what the purpose of the different blocks in Figure 4 is.

(1 p) (b) Give values for L and M such that the output signal $y(n)$ is sampled at a rate of 32 kHz. Motivate your answer.

(2 p) (c) Give an expression for the upsampled signal $v(n)$ as a function of the input signal $x(n)$ in the frequency domain and give a sketch of the spectrum of $v(n)$ both as a function of the normalised angular frequency ω (dimensionless) and the frequency f expressed in cycles/sec (or, equivalently, Hertz (Hz)).

(1 p) (d) What are the specifications of the filter $H(z)$ in terms of pass, stop, and transission band?

(2 p) (e) Give an expression for the signals $w(n)$ and $y(n)$ as a function of the input signal $x(n)$ in the frequency domain and give a sketch of the spectra of w and y both as a function of the normalised angular frequency ω (dimensionless) and the frequency f expressed in cycles/sec (or, equivalently, Hertz (Hz)).