Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Circuits and Systems Group

EE2S11 SIGNALS AND SYSTEMS

Final exam, 29 January 2021, 13:30–15:50
Block 1 (13:30-14:30)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted.
Upload answers during 14:25–14:40

This block consists of three questions (20 points); more than usual, and this will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.

Question 1 (9 points)

(a) Let $x[n] = [\cdots, 0, \boxed{1}, 3, 2, 0, \cdots]$, where the 'box' denotes the value for $n = 0$. Determine the convolution $y[n] = x[n] * x[-n]$.

(b) Let $x[n] = 2^{-n-2}u[-n-2]$. Determine the z-transform, also specify the ROC.

Hint: you could first make a plot of $x[n]$.

(c) Let

$$H(z) = \frac{1 - \frac{3}{2}z^{-2}}{(1 - \frac{1}{2}z^{-1})(1 - 3z^{-1})}\,.$$

Draw a pole-zero plot, and determine $h[n]$ for (c1) ROC: $|z| < \frac{1}{2}$; (c2) ROC: $\frac{1}{2} < |z| < 3$; (c3) ROC: $|z| > 3$.

(d) Let $x[n] = [\cdots, 0, \boxed{1}, 3, 1, 0, \cdots]$. Determine the DTFT $X(e^{j\omega})$, also determine and give plots of the amplitude spectrum and the phase spectrum.

Solution

(a) To avoid confusion, write $r[n] = x[-n] = [\cdots, 0, 2, 3, \boxed{1}, 0, \cdots]$. The convolution is $y[n] = \sum_k r[k]x[n-k]$, where $k = -2, -1, 0$, hence,

$$\begin{array}{lll}
k=-2: & 2\cdot x[n+2] = & [\cdots,0,2,6,\boxed{4},0,0,0,\cdots] \\
k=-1: & 3\cdot x[n+1] = & [\cdots,0,0,3,\boxed{9},6,0,0,\cdots] \\
k=0: & 3\cdot x[n] = & [\cdots,0,0,0,\boxed{1},3,2,0,\cdots] \\
\hline
 & y[n] = & [\cdots,0,2,9,\boxed{14},9,2,0,\cdots]
\end{array}$$

(b) The response is anticausal, stops at $n = -2$. Shifting to the origin gives $2^{-n}u[-n]$, we will need to take into account an 'advance' z^2.

$$X(z) = z^2 \sum_{n=-\infty}^{0} (2z)^{-n} = z^2 \sum_{n=0}^{\infty} (2z)^n = \frac{z^2}{1-2z}\,, \qquad \text{ROC: } |z| < \frac{1}{2}\,.$$

(c) Make proper and do a partial fraction expansion: write as

$$H(z) = A + \frac{B}{1 - \frac{1}{2}z^{-1}} + \frac{C}{1 - 3z^{-1}}$$

where it follows that $A = -1$, $B = 1$, $C = 1$.

(c1): Anticausal response. Rewrite

$$H(z) = -1 - \frac{2z}{1 - 2z} - \frac{\frac{1}{3}z}{1 - \frac{1}{3}z}$$

$$\begin{aligned} h[n] &= -\delta[n] - 2(2)^{-n-1}u[-n-1] - \frac{1}{3}\left(\frac{1}{3}\right)^{-n-1} u[-n-1] \\ &= -\delta[n] - 2^{-n}u[-n-1] - 3^n u[-n-1] \end{aligned}$$

(c2): Mixed causality response (2nd term causal, 3rd term anticausal):

$$H(z) = -1 + \frac{1}{1 - \frac{1}{2}z^{-1}} - \frac{\frac{1}{3}z}{1 - \frac{1}{3}z}$$

$$\begin{aligned} h[n] &= -\delta[n] + \left(\frac{1}{2}\right)^n u[n] - \frac{1}{3}\left(\frac{1}{3}\right)^{-n-1} u[-n-1] \\ &= -\delta[n] + \left(\frac{1}{2}\right)^n u[n] - 3^n u[-n-1] \end{aligned}$$

(c3): Causal response:

$$H(z) = -1 + \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - 3z^{-1}}$$

$$h[n] = -\delta[n] + \left(\frac{1}{2}\right)^n u[n] + 3^n u[n]$$

In each of the above cases, there are several alternative ways to write the answer.

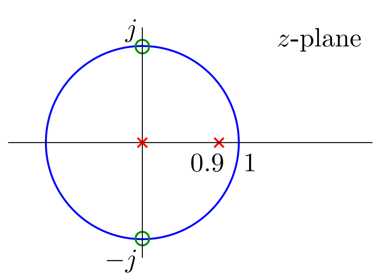
(d)

$$X(e^{j\omega}) = 1 + 3e^{-j\omega} + e^{-j2\omega} = e^{-j\omega}(3 + 2\cos(\omega))$$

Amplitude spectrum: $|X(e^{j\omega})| = 3 + 2\cos(\omega)$. Phase spectrum: $\angle(\omega) = -\omega$.

Question 2 (5 points)

Consider the pole-zero plot of a discrete-time causal filter with transfer function $H(z)$:



(a) Determine $H(z)$, up to an amplitude scale factor A.

(b) Suppose that we know that $h[0] = 2$. Determine A.

(c) Based on the pole-zero locations, contruct a sketch of the magnitude spectrum $|H(e^{j\omega})|$. Clearly indicate relevant values on the ω-axis.

(d) Specify the ROC. Is this a stable filter?

Solution

(a)

$$H(z) = A\frac{(1 - jz^{-1})(1 + jz^{-1})}{(1 - 0\,z^{-1})(1 - 0.9\;z^{-1})} = A\frac{1 + z^{-2}}{1 - 0.9z^{-1}}$$

which is consistent with a pole at zero. Alternatively,

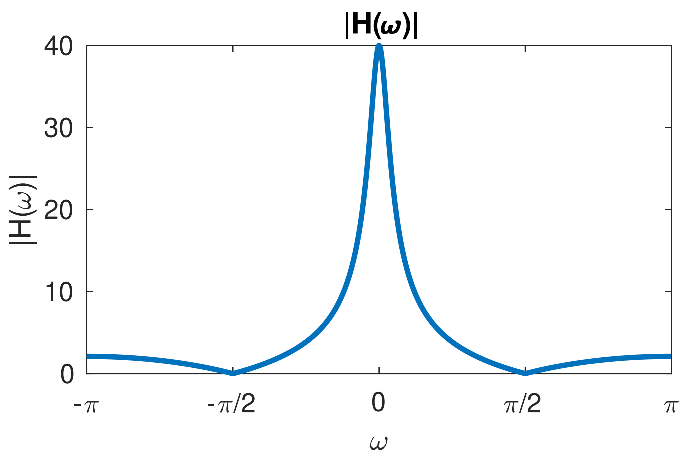
$$H(z) = A\frac{z^2 + 1}{(z - 0.9)z}$$

(b) initial value theorem:

$$h[0] = \lim_{z\to\infty} H(z) = A$$

Hence $A = 2$.

(c) Use phasors. For the amplitude response, the pole at $z = 0$ is irrelevant. The pole at $z = 1$ gives a large peak at $\omega = 0$. The zero locations on the unit circle give zero crossings in the amplitude response. At $z = -1$ ($\omega = \pm\pi$), the response is low. Calculate: $H(z = 1) = A \cdot 2/0.1 = 40$; $H(z = -1) = A \cdot 2/1.9 \approx 2.1$. The amplitude response is an even function.

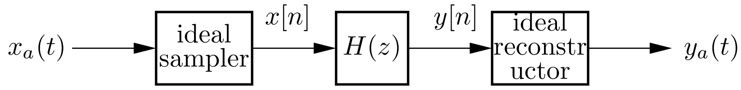


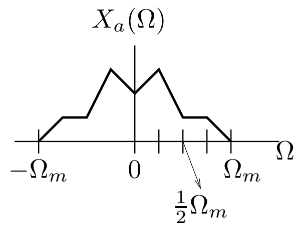
(d) ROC: $|z| > 0.9$. Stable, because the unit circle is in the ROC (or: all poles are within the unit circle).

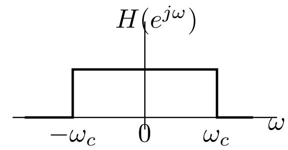
Question 3 (6 points)

A continuous-time signal $x_a(t)$ has a spectrum $X_a(\Omega)$ as indicated below. It is sampled at the Nyquist rate (resulting in $x[n]$), passed through a lowpass filter with frequency response $H(e^{j\omega})$ (resulting in $y[n]$), and reconstructed using an ideal DAC (which includes an ideal interpolation filter). The output signal is $y_a(t)$.

The cut-off frequency of the lowpass filter is $\omega_c = \Omega_m T/2$, where T is the sample period.







(a) Relate T to Ω_m.

(b) Draw the spectra $X(\omega)$, $Y(\omega)$, and $Y_a(\Omega)$. Clearly mark the relevant values on the frequency axis.

(c) Suppose now that we sample at twice the Nyquist rate. Again draw the spectra $X(\omega)$, $Y(\omega)$, and $Y_a(\Omega)$.

Solution

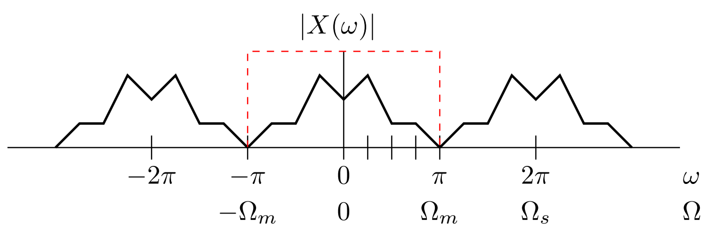
(a) The signal is sampled at Nyquist. Hence,

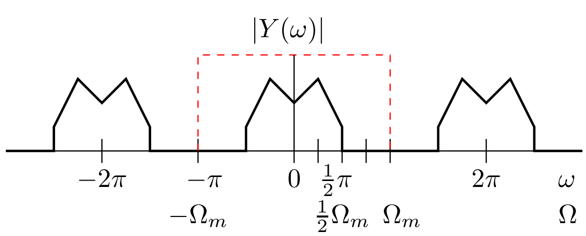
$$F_s = \frac{1}{T} = \frac{2\Omega_m}{2\pi} \qquad \Rightarrow \qquad T = \frac{\pi}{\Omega_m}$$

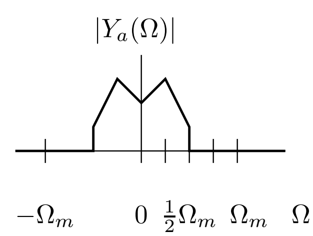
(b)

$$\omega_c = \frac{\Omega_m T}{2} = \frac{1}{2}\pi$$

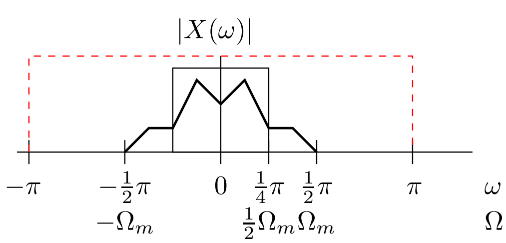
$X(\omega)$ is periodic; note on the frequency axis the relation to $X_a(\Omega)$. $Y(\omega)$ is lowpass filtered but also periodic (the red dashed box indicates the fundamental interval). $Y_a(\Omega)$ is not periodic anymore.

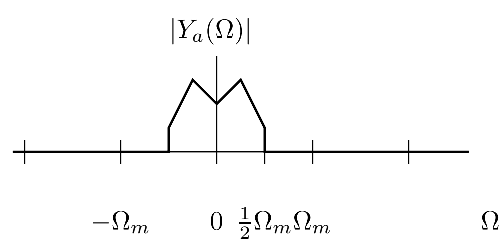






(c) Now, $\omega_c = \frac{1}{4}\pi$, but also the mapping $\Omega \to \omega$ changes:





It follows that the analog signal $Y_a(\Omega)$ is the same as before.

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Block 2 (14:50-15:50)

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Upload answers during 15:45–16:00

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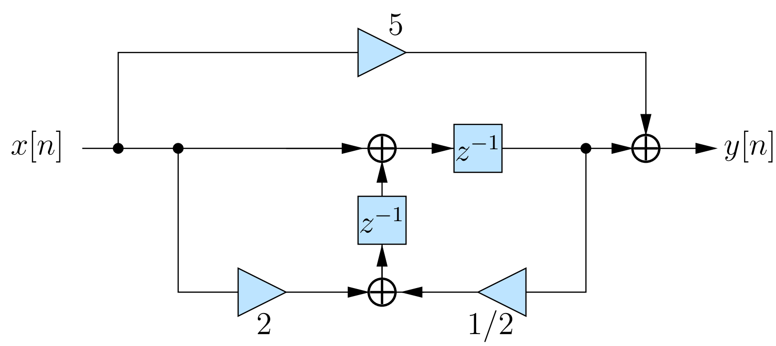
Question 4 (6 points)

Consider the following realization of a causal system:



(a) Determine the transfer function $H(z)$.

(b) What is the difference equation implemented by this realization?

(c) Is this a stable realization? (motivate)

(d) Is this a minimal realization? (motivate)

(e) Draw the "Direct form no. II" realization of the filter and also specify the coefficients.

Solution

(a) Let $P(z)$ be the input of the top delay element, then

$$\begin{cases} P(z) &= \frac{1}{2}z^{-2}P(z) + X(z) + 2z^{-1}X(z) \\ Y(z) &= 5X(z) + z^{-1}P(z) \end{cases}$$

$$\begin{cases} P(z) &= \frac{1+2z^{-1}}{1-\frac{1}{2}z^{-2}}X(z) \\ Y(z) &= \left(5 + z^{-1}\frac{1+2z^{-1}}{1-\frac{1}{2}z^{-2}}\right)X(z) \end{cases}$$

$$\begin{aligned} H(z) &= \frac{5(1-\frac{1}{2}z^{-2}) + z^{-1}(1+2z^{-1})}{1-\frac{1}{2}z^{-2}} \\ &= \frac{5 + z^{-1} - \frac{1}{2}z^{-2}}{1-\frac{1}{2}z^{-2}} \end{aligned}$$

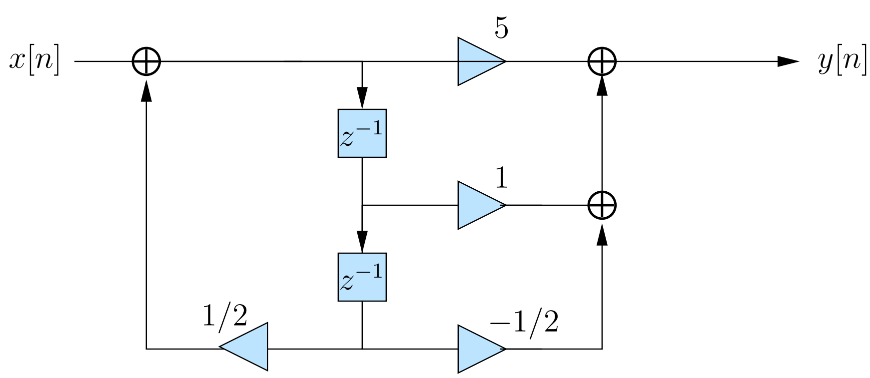
(b)

$$y[n] - \frac{1}{2}y[n-2] = 5x[n] + x[n-1] - \frac{1}{2}x[n-2]$$

(c) Stable, the two poles are $p_{1,2} = 1/\sqrt{2}$, within the unit circle.

(d) Minimal, 2nd order transfer function, and two delays are used.

(e)



Question 5 (9 points)

A normalized second-order analog low-pass filter (Butterworth filter) is given by

$$H_a(s) = \frac{1}{s^2 + \sqrt{2}\,s + 1}.$$

The 3-dB cutoff frequency for this template is $\Omega_c = 1$ rad/s.

We are asked to design a second-order digital *high-pass* filter $G(z)$ with

Passband frequency:	$\omega_p = 1$ rad
Passband damping:	1 dB
Stopband frequency:	$\omega_s = 0.5$ rad

We will first design an analog 2nd order high-pass filter $G_a(s)$ and then apply the bilinear transform.

(a) From the given specifications, what are the passband and stopband frequencies for the analog high-pass filter?

(b) Based on $H_a(s)$, what is the corresponding power spectrum $|H_a(j\Omega)|^2$?

(c) What frequency transformation is needed to transform $|H_a(j\Omega)|^2$ to a template $|G_a(j\Omega)|^2$ for the analog 2nd order high-pass filter, which involves design parameters ϵ and Ω_p ?

(d) What is the corresponding template high-pass filter $G_a(s)$?

(e) Compute the unknown parameters:

What is $|G_a(j\Omega)|^2$ and $G_a(s)$ that satisfies the specifications?

(f) What is the resulting digital high-pass filter $G(z)$ that satisfies the specifications?

(g) How much damping in the stopband is achieved? (specify in dB)

Solution

(a) $\Omega_p = \tan(\omega_p/2) = 0.5463$, $\Omega_s = \tan(\omega_s/2) = 0.2553$.

(b)

$$|H_a(j\Omega)|^2 = H(s)H(-s)\big|_{s=j\Omega} = \frac{1}{-\Omega^2 + j\sqrt{2}\Omega + 1} \cdot \frac{1}{-\Omega^2 - j\sqrt{2}\Omega + 1} = \frac{1}{(1-\Omega^2)^2 + 2\Omega^2} = \frac{1}{1+\Omega^4}$$

which indeed corresponds to a Butterworth of order 2.

(c) We want to obtain a filter of the form

$$|G_a(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 \left(\frac{\Omega_p}{\Omega}\right)^4}$$

Comparing to (b), the transformation we need is

$$\Omega \to \sqrt{\epsilon}\frac{\Omega_p}{\Omega}, \qquad s \to \sqrt{\epsilon}\frac{\Omega_p}{s}$$

(Instead of $\sqrt{\epsilon}$, we could use another scale, e.g. introduce a parameter α, as long as we take that into account into the resulting template for $|G_a(j\Omega)|^2$.)

(d) Apply the transformation to $H_a(s)$:

$$G_a(s) = \frac{1}{\epsilon\frac{\Omega_p^2}{s^2} + \sqrt{2\epsilon}\frac{\Omega_p}{s} + 1}$$

(e) In the equation for $|G_a(j\Omega)|^2$, fill in $\Omega = \Omega_p$:

$$|G_a(j\Omega_p)|^2 = \frac{1}{1+\epsilon^2} = 10^{-1/10} = 0.7943$$

$$\epsilon = \sqrt{1/0.7943 - 1} = 0.5089\,.$$

Hence

$$G_a(s) = \frac{s^2}{0.1519 + 0.5511\,s + s^2}$$

(f) Substitute the bilinear transform,

$$s \to \frac{1 - z^{-1}}{1 + z^{-1}}$$

resulting in

$$\begin{aligned}
G(z) &= \frac{\frac{(1-z^{-1})^2}{(1+z^{-1})^2}}{0.1519 + 0.5511\,\frac{1-z^{-1}}{1+z^{-1}} + \frac{(1-z^{-1})^2}{(1+z^{-1})^2}} \\
&= \frac{(1-z^{-1})^2}{0.1519(1+z^{-1})^2 + 0.5511\,(1-z^{-1})(1+z^{-1}) + (1-z^{-1})^2} \\
&= \frac{(1-z^{-1})^2}{0.1519 + 2\cdot 0.1519z^{-1} + 0.1519z^{-2} + 0.5511 - 0.5511z^{-2} + 1 - 2z^{-1} + z^{-2}} \\
&= \frac{(1-z^{-1})^2}{1.7030 - 1.6962z^{-1} + 0.6008z^{-2}}
\end{aligned}$$

(g) Most reliable/straightforward is to fill in Ω_s in the formula for $|G_a(j\Omega)|^2$:

$$|G_a(j\Omega_s)|^2 = \frac{1}{1 + \epsilon^2 \left(\frac{\Omega_p}{\Omega_s}\right)^4} = \frac{1}{1 + (0.5089)^2 \left(\frac{0.5463}{0.2553}\right)^4} = 0.1555$$

Thus, the damping is $10\log(0.1555) = -8.1$ dB.

(You could also take $G(z)$, insert $z = e^{j\omega_s}$, and compute the norm of the result. You'll have to deal with complex numbers.)

Question 6 (5 points)

(a) Determine the Fourier transform of

$$x(t) = \cos(\Omega_0 t)\sin(\Omega_1 t)\,.$$

(b) A periodic signal $x(t)$ has a Fourier series

$$x(t) = \sum_{k=1}^{\infty} \frac{2}{k^2}\cos(3kt/2)\,.$$

Compute the Fourier transform, $X(\Omega)$.

(c) Use the duality theorem to prove the following Fourier transform result:

$$x(t) = \frac{1}{t^2 + a^2}\,, \quad a > 0 \qquad \leftrightarrow \qquad X(\Omega) = \frac{\pi}{a}e^{-a|\Omega|}$$

Solution

(a) Using the multiplication property,

$$\begin{aligned}
X(\Omega) &= \frac{\pi^2}{2\pi}\big[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)\big] * (-j)\big[\delta(\Omega - \Omega_1) - \delta(\Omega + \Omega_1)\big] \\
&= \frac{j\pi}{2}\big[\delta(\Omega - \Omega_0 + \Omega_1) + \delta(\Omega + \Omega_0 + \Omega_1) - \delta(\Omega - \Omega_0 - \Omega_1) - \delta(\Omega + \Omega_0 - \Omega_1)\big]
\end{aligned}$$

(b)

$$X(\Omega) = \pi \sum_{k=1}^{\infty} \frac{2}{k^2}[\delta(\Omega - 3k/2) + \delta(\Omega + 3k/2]$$

(c) Start with the LT pairs

$$\begin{array}{lcl} e^{-at}u(t) & \leftrightarrow & \frac{1}{s+a} \\ e^{at}u(-t) & \leftrightarrow & \frac{1}{-s+a} \\ e^{-a|t|} & \leftrightarrow & \frac{1}{s+a} + \frac{1}{-s+a} = \frac{2a}{a^2-s^2} \end{array}$$

Thus, we have the FT pair ($s = j\Omega$)

$$y(t) = e^{-a|t|} \qquad \leftrightarrow \qquad Y(\Omega) = \frac{2a}{a^2 + \Omega^2}$$

The duality theorem gives then

$$Y(t) = \frac{2a}{a^2 + t^2} \qquad \leftrightarrow \qquad 2\pi y(-\Omega) = 2\pi e^{-a|\Omega|}$$

Finally, by rescaling we obtain then that

$$\frac{1}{a^2 + t^2} \qquad \leftrightarrow \qquad \frac{\pi}{a} e^{-a|\Omega|}$$