Delft University of Technology
Faculty of Electrical Engineering, Mathematics and Computer Science

Partial exam EE2S11 Signals and Systems
Part 2: 1 February 2019, 13:30–15:30

Closed book; two sides of handwritten notes permitted

This exam consists of five questions (40 points)

Question 1 (12 points)

a) Given the signal $x[n] = [\cdots, 0, \boxed{0}, 1, 2, 0, \cdots]$.

Determine $y[n] = x[n] * x[-n]$ using the convolution sum (in time-domain).

b) Also determine $y[n]$ in a) via the z-transform (do you obtain the same result?).

c) Given $x[n] = (n-1)\,u[n]$. Determine $H(z)$, also specify the ROC.

d) Given $X(z) = \dfrac{4z}{(z-1)(z+0.25)}$, $\qquad$ ROC $= \{|z| > 1\}$.

Determine $x[n]$ using the inverse z-transform.

e) Let $x[n] = u[n+2] - u[n-3]$. Determine the DTFT $X(e^{j\omega})$.

f) Suppose the DTFT of a signal $x[n]$ is $X(e^{j\omega})$. What is the DTFT of $\cos(3n) \cdot x[n]$?

Solution

a) Define $r[n] = x[-n] = [\cdots, 2, 1, \boxed{0}, \cdots]$. Using $y[n] = x[n] * r[n] = \sum_{k=-\infty}^{\infty} x[k]r[n-k]$, we find

$$\begin{array}{rcl}
(k=1)\; 1\cdot r[n-1] & = & [\cdots,\; 0,\; 2\; \boxed{1},\; 0,\; 0,\; \cdots] \\
(k=2)\; 2\cdot x[n-2] & = & [\cdots,\; 0,\; 0,\; \boxed{4},\; 2,\; 0,\; \cdots] \\
\hline
y[n] & = & [\cdots,\; 0,\; 2,\; \boxed{5},\; 2,\; 0,\; \cdots]
\end{array}$$

You can also directly evaluate $y[n] = \sum x[k]x[-n+k]$ in this way but be careful not to get confused. By replacing $k \to n+k$, you can also write $y[n] = \sum x[k]x[n+k]$.

b) Since $X(z) = z^{-1} + 2z^{-2}$, we find

$$Y(z) = X(z) \cdot X(z^{-1}) = (z^{-1} + 2z^{-2})(z + 2z^2) = 2z + 5 + 2z^{-1}$$

Hence $y[n] = [\cdots, 0, 2, \boxed{5}, 2, 0, \cdots]$.

c) Using the table,

$$X(z) = \frac{z^{-1}}{(1-z^{-1})^2} - \frac{1}{1-z^{-1}} = \frac{z^{-1} - (1-z^{-1})}{(1-z^{-1})^2} = \frac{2z^{-1} - 1}{(1-z^{-1})^2}$$

ROC: $\{|z| > 1\}$.

d)

$$X(z) = \frac{4z^{-1}}{(1-z^{-1})(1+0.25z^{-1})} = \frac{16/5}{1-z^{-1}} - \frac{16/5}{1+0.25z^{-1}}$$

Check the ROC: both terms are causal. Hence

$$x[n] = \frac{16}{15}u[n] - \frac{16}{5}\left(-\frac{1}{4}\right)^n u[n]\,.$$

Alternatively, we find $x[n] = \frac{16}{5}u[n-1] + \frac{4}{5}\left(\frac{-1}{4}\right)^{n-1} u[n-1]$.

e) Write $x[n] = [\cdots, 0, 1, 1, \boxed{1}, 1, 1, 0, \cdots]$, so that

$$X(e^{j\omega}) = e^{j2\omega} + e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega} = 1 + 2\cos(\omega) + 2\cos(2\omega)$$

This can also be obtained from the z-transform, but notice the pole-zero cancellation. Alternatively, the z-transform of a shifted pulse gives

$$X(e^{j\omega}) = \frac{\sin(\frac{5}{3}\omega)}{\sin(\frac{1}{2}\omega)}$$

which is actually the same.

f) Use the modulation property of the DTFT:

$$Y(e^{j\omega}) = \frac{1}{2}\left(X(\omega - 3) + X(\omega + 3)\right)$$

Question 2 (6 points)

A causal system is specified by the transfer function

$$H(z) = \frac{z^2 + 1}{(z - 0.9)(z + 0.9)}\,.$$

a) Determine all poles and zeros of the system and draw a pole-zero plot.

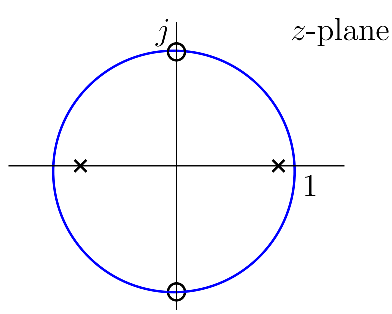
b) What is the ROC?

c) Is this a stable system?

d) Sketch the amplitude spectrum $|H(e^{j\omega})|$, also indicate values on the frequency axis.

Solution

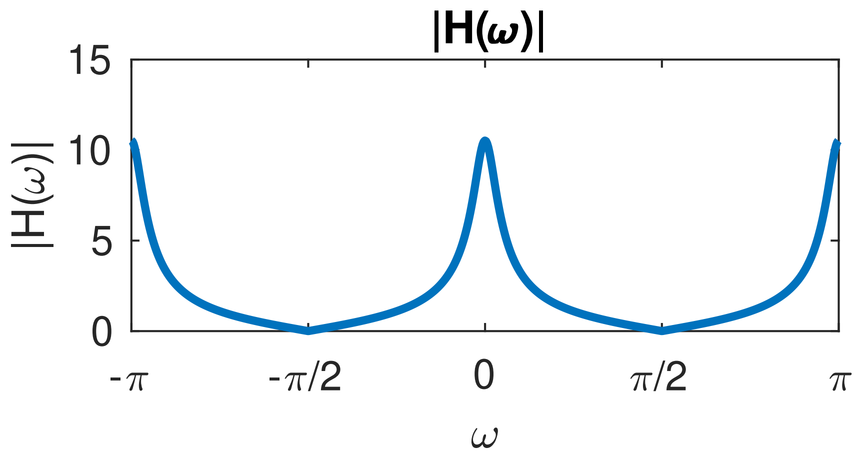
a) Poles: $z = 0.9$, $z = -0.9$; zeros: $z = \pm j$.



b) Causal, hence ROC: $\{|z| > 0.9\}$.

c) Unit circle in ROC: stable. (Or: causal and no poles on or outside the unit circle.)

d) Use fasors:

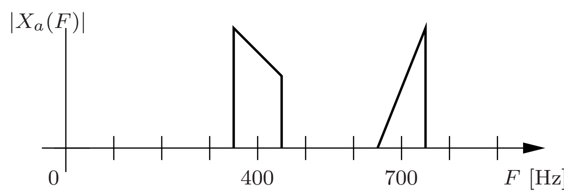


Peaks at $\omega = 0, \pi$. At the peaks the function is flat (hard to see in the matlab plot, but you can indicate this more clearly in the sketch). Evaluate $|H(\omega = 0)| = \frac{2}{0.1 \cdot 1.9} = 10.530$.

Zero at $\omega = \pi/2$. At the zeros, the function is not flat (similar to $|\cos(\omega)|$ at a zero crossing of $\cos(\omega)$).

Question 3 (7 points)

A real-valued continuous-time signal $x_a(t)$ has frequency components around 400 Hz and 700 Hz, as shown in the figure (the bands are 100 Hz wide).



a) At which frequency should we at least sample to avoid loss of information or distortion?

b) The signal is sampled at $F_s = 1000$ Hz, resulting in $x[n]$, no filtering is applied. Draw the amplitude spectrum of $x[n]$, also indicate the frequency axis for ω and relate it to the corresponding frequencies in Hz.

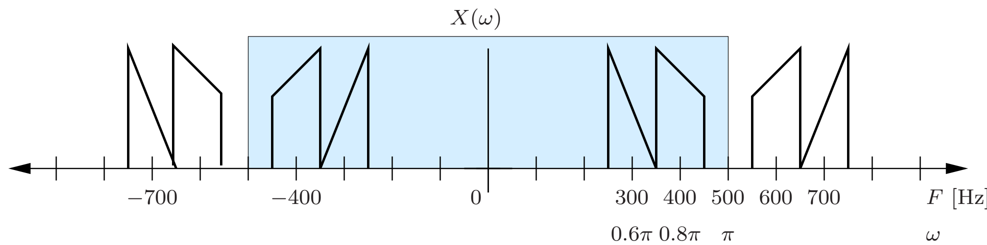
c) Is it possible to reconstruct the original signal $x_a(t)$ from the sampled signal? (Motivate your answer.)

d) Discuss what happens if $F_s = 1100$ Hz.

Solution

a) 1500 Hz (Nyquist).

b) The original signal is real: the complete spectrum also has (mirrored) components at -400 and -700 Hz. Due to aliasing, the component at 700 Hz returns at $700 \pm k \cdot 1000$, and the component at -700 Hz at $-700 \pm k \cdot 1000$, e.g. at 300 Hz. The component at 400 Hz returns at -600 Hz, and the one at -400 Hz at 600 Hz. Watch out for the mirroring.



c) In this case: yes.

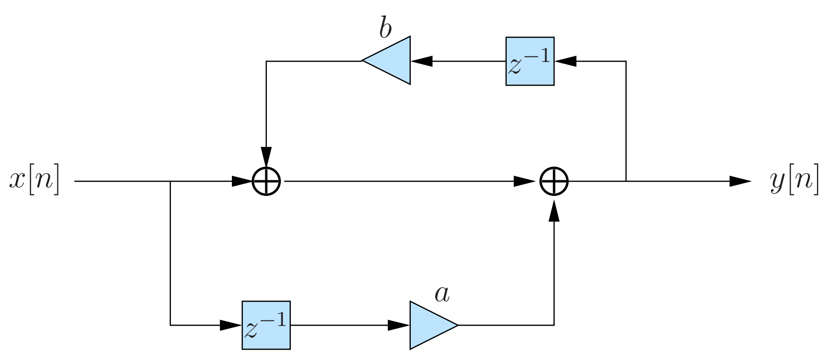
Aliasing is present: not a priori possible to recover the signals. But if you know beforehand which frequency blocks were present, you can isolate them using filtering (because after sampling they did not overlap) and reconstruct each of them separately.

You can do this e.g. by a D/A conversion, followed by an analog bandpass filter $H_a(F)$ which isolates exactly the desired frequency bands (350-450 Hz en 650-750 Hz).

d) In this case we also have aliasing, but now the components overlap after aliasing ($-700 + 1100 = 400$), and reconstruction is not possible anymore.

Question 4 (5 points)

a) Determine the transfer function $H(z)$ of the following realization:



b) Is this a minimal realization? (Why?)

c) Draw the "Direct form no. II" realization and also specify the coefficients.

Solution

a) Insert additional variables: call $P(z)$ the output of the multiplier a, and $Q(z)$ the output of the multiplier b.

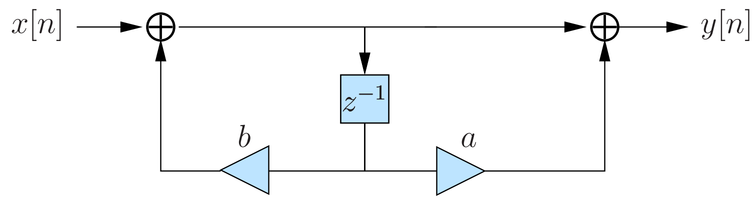
$$\begin{aligned} P(z) &= az^{-1}X(z) \\ Q(z) &= bz^{-1}Y(z) \\ Y(z) &= X(z) + P(z) + Q(z) \end{aligned}$$

so that $Y(z) = X(z) + P(z) + Q(z) = X(z) + az^{-1}X(z) + bz^{-1}Y(z)$, and

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + az^{-1}}{1 - bz^{-1}}$$

b) Not minimal: the number of delay elements is 2 but the filter is first order.

c) Direct form II realization:



Question 5 (10 points)

Design a first-order digital lowpass filter $H(z)$ satisfying the following specifications:

Passband frequency: $\omega_p = 0.3\pi$,
Damping outside the passband: at least 10 dB

Use the bilinear transform and base your design on an analog Butterworth filter.

a) What is the passband frequency in the analog frequency domain?

b) What is the generic form of $|H_a(j\Omega)|^2$ of a first-order analog Butterworth filter? What is the corresponding $H_a(s)$?

c) Determine $|H_a(j\Omega)|^2$ that meets the specifications.

d) What is the corresponding analog filter $H_a(s)$ that meets the specifications?

e) What is the desired digital filter $H(z)$?

f) Demonstrate (verify) that the resulting $H(z)$ meets the specifications.

Solution

a) $\Omega_p = \tan(\omega_p/2) = 0.5095$.

b) $|H_a(\Omega)|^2 = \dfrac{1}{1+\epsilon^2(\Omega/\Omega_p)^2}$.

With $s = j\Omega \Leftrightarrow \Omega = -js$ we find $H_a(s)H_a(-s) = \dfrac{1}{1-\epsilon^2(s/\Omega_p)^2}$ and $H(s) = \dfrac{1}{1+\epsilon s/\Omega_p}$.

c) Determine ϵ from the requirement on the damping at $\omega_p = 0.3\pi$. From $|H(\Omega_p)|^2 = 10^{-10/10} = 0.1$ (damping 10 dB) it follows that

$$\frac{1}{1+\epsilon^2} = 0.1 \qquad \Rightarrow \qquad \epsilon = 3\,,$$

$$|H_a(j\Omega)| = \frac{1}{1+\epsilon^2(\Omega/\Omega_p)^2} = \frac{1}{1+34.7\,\Omega^2}\,.$$

d) $H_a(s) = \dfrac{1}{1+\epsilon/\Omega_p \cdot s} = \dfrac{1}{1+5.88\cdot s}$.

e) Use the bilinear transform:

$$s \to \frac{1-z^{-1}}{1+z^{-1}}\,.$$

This results in

$$H(z) = \frac{1}{1+5.88\frac{1-z^{-1}}{1+z^{-1}}} = \frac{1}{6.88}\cdot\frac{1+z^{-1}}{1-0.7096\cdot z^{-1}}\,.$$

f) Verify:

$$|H(\omega=0)| = |H(z^{-1}=1)| = 1$$
$$|H(\omega=\pi)| = |H(z^{-1}=-1)| = 0$$
$$|H(\omega=0.3\pi)|^2 = |H(z^{-1}=0.59-j0.81)|^2 = \tfrac{1}{(6.88)^2}\tfrac{(1+0.59)^2+(0.81)^2}{(1-0.7096\cdot 0.59)^2+(0.7096\cdot 0.81)^2} = \cdots = 0.1\,.$$