Partial exam EE2S11 Signals and Systems
Part 2: 2 February 2018, 13:30–15:30

Closed book; two sides of handwritten notes permitted

This exam consists of five questions (40 points)

Question 1 (10 points)

a) Given the signals $x[n] = \delta[n+1] - 2\delta[n-2]$ and $h[n] = [\cdots, 0, \boxed{1}, 3, 2, 0, \cdots]$.
Determine $y[n] = x[n] * h[n]$ using the convolution sum (in time-domain).

b) Given $x[n] = -(\frac{1}{2})^{-n-1}u[-n-1]$. Determine $X(z)$ and also specify the ROC.

c) Given $X(z) = \dfrac{4z^{-1}}{(1-z^{-1})(1+0.25z^{-1})}$, ROC $= \{|z| < .25\}$.
Determine $x[n]$ using the inverse z-transform.

d) Given $x[n] = a^n u[n]$ with $|a| < 1$. Use the z-transform to determine $E = \sum_{n=0}^{\infty}(x[n])^2$.

e) Suppose the DTFT of a signal $x[n]$ is $X(e^{j\omega})$. What is the DTFT of $x[n-3]$?

f) Let $h[n]$ be the impulse response of an ideal low-pass filter with cut-off frequency at 0.4π. Let the impulse response of a new filter be $h_1[n] = (-1)^n h[n]$.
Determine the frequency response $H_1(e^{j\omega})$ in terms of $H(e^{j\omega})$, and give a sketch of it.

Solution

a) Compute $y[n] = x[n] * h[n] = \displaystyle\sum_{k=0}^{\infty} x[k]h[n-k]$: the only nonzero terms occur for $k=-1$ and $k=2$.

$x[-1]h[n+1]$:	$[\cdots 0$	1	$\boxed{3}$	2	0	0	0	$0 \cdots]$
$x[2]h[n-2]$:	$[\cdots 0$	0	$\boxed{0}$	0	-2	-6	-4	$0 \cdots]$
$y[n]$:	$[\cdots 0$	1	$\boxed{3}$	2	-2	-6	-4	$0 \cdots]$

b)

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} = -\sum_{n=-\infty}^{\infty} (\frac{1}{2})^{-n-1}u[-n-1]z^{-n} = -\sum_{n=-\infty}^{\infty} (\frac{1}{2})^{n-1}u[n-1]z^{n} \\ &= -\sum_{n=1}^{\infty}(\frac{1}{2})^{n-1}z^n = -z\sum_{n=0}^{\infty}(\frac{1}{2})^n z^n = \frac{-z}{1-\frac{1}{2}z} = \frac{2z}{z-2} \end{aligned}$$

ROC: $|z| < 2$ (the signal is anti-causal)

c)

$$X(z) = \frac{4z^{-1}}{(1-z^{-1})(1+0.25z^{-1})} = \frac{16/5}{1-z^{-1}} - \frac{16/5}{1+0.25z^{-1}}$$

Check the ROC: both terms are anti-causal. Hence we write

$$X(z) = \frac{-16/5\,z}{1-z} - \frac{16/5\,z}{z+0.25} = \frac{-16/5\,z}{1-z} - \frac{64/5\,z}{1+4z}$$

$$x[n] = -\frac{16}{5}u[-n-1] - \frac{64}{5}(-4)^{-n-1}u[-n-1]$$

d) Write $e[n] = (x[n])^2 = a^{2n}u[n]$, then

$$E(z) = \sum e[n]z^{-n} = \frac{1}{1 - a^2 z^{-1}}$$

Evaluate for $z = 1$ gives $E = \sum(x[n])^2 = \frac{1}{1-a^2}$. This could in fact be obtained without using the z-transform.

Alternative (perhaps too complicated...): define $e[n]$ as

$$e[n] = x[n] * x[-n]$$

then $e[0] = \sum(x[n])^2$. (The sequence $e[n]$ is known as a correlation sequence.) Take the z-transform:

$$E(z) = \frac{1}{1 - az^{-1}} \cdot \frac{1}{1 - az}$$

We need to evaluate the term with z^0 from $E(z)$ (note that it is a 2-sided sequence). Thus

$$\begin{aligned} E(z) &= \frac{z}{z-a} \cdot \frac{1}{1-az} = \frac{A}{z-a} + \frac{B}{1-az} = \cdots \\ &= \frac{a}{1-a^2}\frac{1}{z-a} + \frac{1}{1-a^2}\frac{1}{1-az} \\ &= \frac{a}{1-a^2}\frac{z^{-1}}{1-az^{-1}} + \frac{1}{1-a^2}\frac{1}{1-az} \end{aligned}$$

The first term is a polynomial in z^{-1} starting with z^{-1}, the second is a polynomial in z starting with z^0. The term in $E(z)$ with z^0 is found as

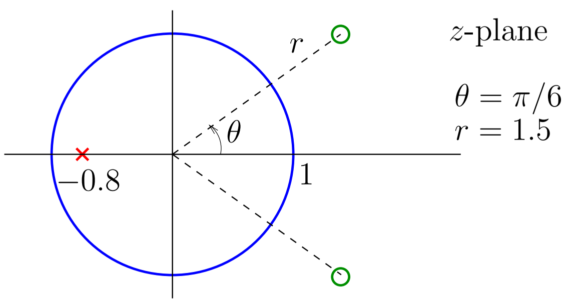
$$e[0] = \lim_{z=0} \frac{1}{1-a^2}\frac{1}{1-az} = \frac{1}{1-a^2}$$

e) This corresponds to a delay of 3 samples. The z-transform would be $z^{-3}X(z)$, hence the answer is $e^{-3j\omega}X(e^{j\omega})$.

f) Use the modulation property: $H_1(e^{j\omega}) = H(e^{j(\omega-\pi)})$. This shift of the spectrum by π transforms a low-pass to a highpass (with cut-off at 0.6π).

Question 2 (8 points)

A causal system $H(z)$ has the following pole-zero diagram:



a) What does the fact that $H(z)$ is a causal system tell you on the ROC of $H(z)$?

b) Specify $H(z)$. (Assume a gain such that $H(z) = 1$ for $z = 1$.)

c) Is this a stable system?

d) Sketch the amplitude spectrum $|H(e^{j\omega})|$, also indicate values on the frequency axis.

e) Give a pole-zero diagram of the inverse system, $G(z) = [H(z)]^{-1}$. Is this a causal stable system?

Solution

a The ROC extends from a circle containing the "largest" pole until infinity, i.e., ROC: $|z| > 0.8$.

b

$$H(z) = G\,\frac{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})}{1 + 0.8z^{-1}} = G\,\frac{1 - 2r\cos(\theta)z^{-1} + r^2 z^{-2}}{1 + 0.8z^{-1}} = G\,\frac{1 - \frac{3}{2}\sqrt{3}z^{-1} + (1.5)^2 z^{-2}}{1 + 0.8z^{-1}}$$

$$H(1) = G\frac{1 - \frac{3}{2}\sqrt{3} + (1.5)^2}{1 + 0.8} = G \cdot 0.3622$$

$$\Rightarrow \qquad G = \frac{1}{0.3622} = 2.7611$$

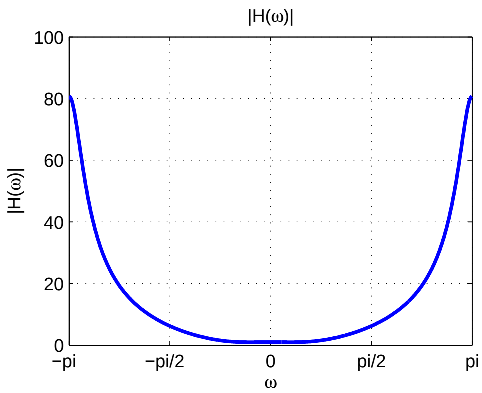
An alternative would be

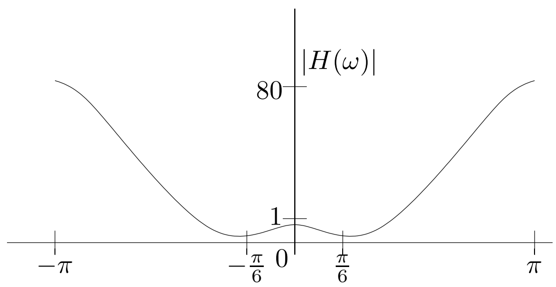
$$H(z) = G\,\frac{(z - re^{j\theta})(z - re^{-j\theta})}{z + 0.8}$$

However, this has a pole at ∞ (equivalently, you can split off a term containing z when making it proper) and that is not compatible with the stated causality.

c Unit circle contained in the ROC (for a causal system equal to: all poles contained in the unit circle): stable

d Here is a matlab plot, and a sketch that better shows the essential features.



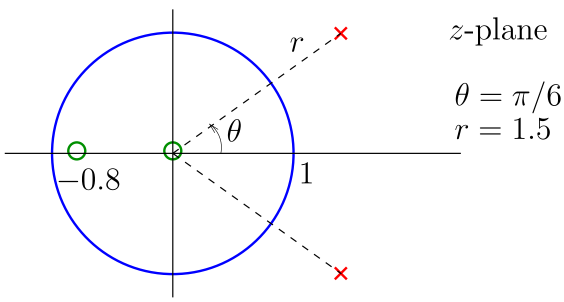


The sketch should show: for $\omega = 0$ the amplitude response is $|H(e^{j\omega})| = 1$, for $\omega = \pm\pi$, the amplitude response is maximal (with flat derivative), for approximately $\omega = \pi/6$, the response is minimal (but not quite zero).

e

$$G(z) = H^{-1}(z) = \frac{1}{G}\frac{1 + 0.8z^{-1}}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})} = \frac{1}{G}\frac{z(z + 0.8)}{(z - re^{j\theta})(z - re^{-j\theta})}$$

Note: there is a zero voor $z = 0$.

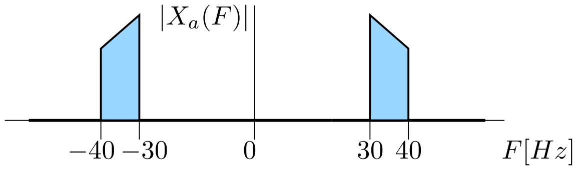


For $G(z)$, we can select an ROC $|z| > 1.5$. In that case, $G(z)$ is causal but not stable (ROC does not contain the unit circle). Alternatively, we select ROC $|z| < 1.5$. In that case, $G(z)$ is anti-causal but stable (ROC contains unit circle).

Question 3 (5 points)

A continuous-time signal $x_a(t)$ has frequencies in the range 30 to 40 Hz. The signal is sampled with period T_s so that we obtain a discrete-time signal $x[n] = x_a(nT_s)$.

The amplitude spectrum $|X_a(F)|$ of $x_a(t)$ is as follows (with F in hertz, using $\Omega = 2\pi F$):



a) What is the Nyquist frequency at which $x_a(t)$ should be sampled to avoid aliasing?

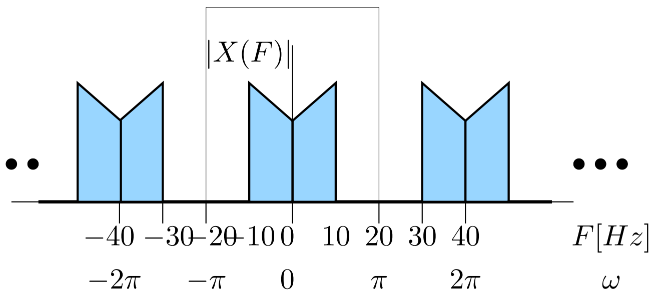
b) We sample at a rate of 40 Hz. Give a drawing of the amplitude spectrum $|X(\omega)|$ of the signal $x[n]$. Also indicate the frequency axis for ω and relate it to the corresponding frequencies in Hz.

c) Is it possible to reconstruct $x_a(t)$ from $x[n]$? If not, why not? If yes, indicate how this could be done. (Assume ideal D/A converters and ideal low-pass filters.)

Solution

a $F_s = 80$ Hz

b With $F_s = 40$ Hz we obtain after sampling the original spectrum, shifted by multiples of ± 40 Hz. The part of the spectrum at 30-40 Hz will also return at -10-0 Hz (in the fundamental interval). The part of -40--30 Hz will also return at 0-10 Hz.

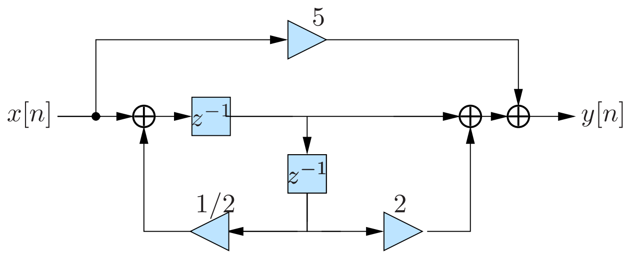


c We can first take an ideal DAC: this will return an analog signal (consisting of delta-spikes) with the same spectrum as the digital signal. Then apply an ideal analog bandpass filter which passes 30-40 Hz.

Note that we cannot first apply the bandpass filter and then do the DAC, because the spectrum of a digital signal is periodic and we cannot filter to keep the band corresponding to 30-40 Hz in the digital domain without also keeping the band from -10 to 10 Hz.

Question 4 (6 points)

a) Determine the transfer function $H(z)$ of the following realization:



b) Is this a minimal realization?

c) Draw the "Direct form no. II" realization and also specify the coefficients.

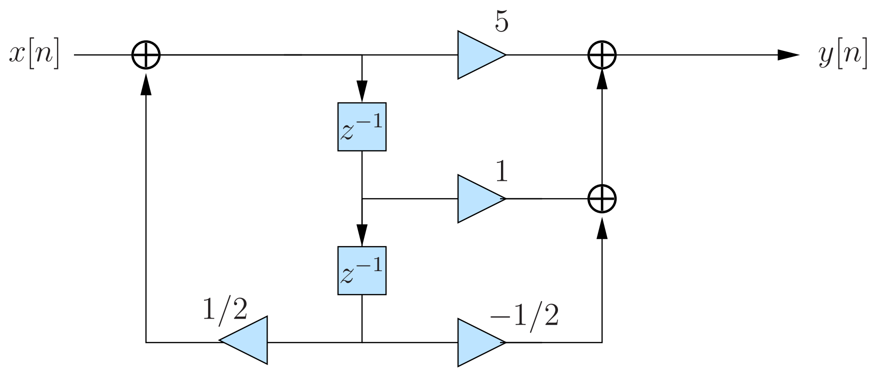
Solution

a

$$H(z) = 5 + \frac{z^{-1} + 2z^{-2}}{1 - 1/2z^{-2}} = \frac{5 + z^{-1} - 1/2z^{-2}}{1 - 1/2z^{-2}}$$

b Yes: 2 delays used for a 2nd order filter.

c



Question 5 (11 points)

We would like to design a *digital high*-pass filter with the following specifications:

Pass-band: starting at 5.0 kHz
Stop-band: below 3.0 kHz
Sample rate: 20 kHz
Ripple in the pass-band : ≤ 0.5 dB
Stop-band damping: ≥ 30 dB

The digital filter is designed by applying the bilinear transform to an analog transfer function.

a) What are the pass-band and stop-band frequencies (in rad) in the digital time-domain?

b) What are the filter specifications in the analog time-domain?

c) Give a template expression for the amplitude response of an n-th order analog low-pass Butterworth filter.

d) What frequency transformation is needed to transform this into an analog high-pass filter? What is the resulting template expression for the analog high-pass filter, $|G(j\Omega)|^2$?

e) Use the design specifications to compute the unknown parameters of $|G(j\Omega)|^2$.

f) It is known that the poles of a low-pass Butterworth filter are all located on a (semi-)circle in the complex s-plane. What can you say about the poles and zeros of the high-pass filter, as a result of the lowpass-to-highpass transformation?

g) Is the bilinear transform suitable for transforming a high-pass analog filter? (Why?)

h) What can you say on the location of the zeros of the digital filter, resulting after the bilinear transformation?

Solution

a)

$$f_p = \frac{5}{20} = \frac{1}{4} \;\Rightarrow\; \omega_p = \frac{\pi}{2} \text{ rad}$$

$$f_s = \frac{3}{20} \;\Rightarrow\; \omega_s = \frac{6\pi}{20} = 0.3\pi \text{ rad}$$

b) Use the bilinear transform: $\omega = 2\arctan(\Omega)$, $\Omega = \tan(\frac{\omega}{2})$:

$$\Omega_p = \tan(\frac{\omega_p}{2}) = 1$$

$$\Omega_s = \tan(\frac{\omega_s}{2}) = 0.5095$$

For the ripples: $\delta_p = 10^{-0.5/20} = 0.9441$, $\delta_s = 10^{-30/20} = 0.0316$.

c) Butterworth: $|H(j\Omega)|^2 = \dfrac{1}{1+\epsilon^2(\Omega/\Omega_p)^{2N}}$.

d) $s \to \frac{\Omega_p^2}{s}$, $\Omega \to \frac{\Omega_p^2}{\Omega}$:

$$|G(j\Omega)|^2 = \frac{1}{1+\epsilon^2(\Omega_p/\Omega)^{2N}}$$

(In this case, $\Omega_p = 1$ which can be inserted to simplify the expressions.)

e) For the passband (evaluate at $\Omega = \Omega_p$), we have:

$$|G(\Omega_p)|^2 = \frac{1}{1+\epsilon^2} = \delta_p^2 \quad\Rightarrow\quad \epsilon = \sqrt{\frac{1}{\delta_p^2} - 1} = 0.3493$$

For the filter order (evaluate at $\Omega = \Omega_s$):

$$|G(\Omega_s)|^2 = \frac{1}{1+\epsilon^2(\Omega_p/\Omega_s)^{2N}} = \delta_s^2 \quad\Rightarrow\quad (\frac{\Omega_p}{\Omega_s})^{2N} = \frac{\frac{1}{\delta_s^2} - 1}{\epsilon^2} =: \frac{\delta^2}{\epsilon^2} \quad\Rightarrow\quad N \geq \frac{\log(\delta/\epsilon)}{\log(\Omega_p/\Omega_s)}$$

Substitution results in $\delta = 31.6070$ and $N \geq 6.6815$, i.e., the filter order is $N \geq 7$.

f) Denote the poles by $s_1, \cdots, s_N$ and let $a = \Omega_p^2$ and $c = 1/(s_1 \cdots s_N)$, then

$$H(s) = \frac{1}{(s-s_1)\cdots(s-s_N)} \quad\Rightarrow\quad G(s) = \frac{1}{(\frac{a}{s}-s_1)\cdots(\frac{a}{s}-s_N)} = c\,\frac{s^N}{(\frac{a}{s_1}-s)\cdots(\frac{a}{s_N}-s)}$$

There are N zeros at $s = 0$ (which ensure that the filter response is exactly zero at $\Omega = 0$), and the poles are still on a semicircle in the left-hand plane

Certainly not correct to say that the poles and zeros swap places: we do not have $G(s) = [H(s)]^{-1}$.

g) Yes; it will transform an analog high-pass filter into a digital high-pass filter. It results in a one-to-one transformation of the analog Ω-axis into the digital ω-axis.

h) The N zeros at $\Omega = 0$ are transformed into N zeros at $\omega = 0$.