Delft University of Technology
Faculty of Electrical Engineering, Mathematics and Computer Science

EE2S11 SIGNALS AND SYSTEMS

Part 1, 13 December 2018, 13:30 - 15:30

Closed book; two sides of one A4 page with handwritten notes permitted. Graphic calculators not permitted.

This exam has four questions (38 points)

Question 1 (10 points)

Evaluate the following integrals:

a)
$$\int_{t=-4}^{4} t^2\left[\delta(t+2)+\delta(t)+\delta(t-5)\right]\,\mathrm{d}t$$

b)
$$\int_{t=-4}^{4} (t^2+2)\left[\delta(t)+3\delta(t-2)\right]\,\mathrm{d}t$$

c)
$$\int_{t=-4}^{4} t^2\delta'(t-2)\,\mathrm{d}t$$

The signal $y(t)$ satisfies the differential equation

$$4y'(t)+8y(t)=12\delta(t)\,, \qquad \text{with } y(0^-)=0\,.$$

d) Determine $y(0^+)$.

e) Find a signal $z(t)$ such that $z(t)=0$ for $t<0$ and $z'(t)=12\delta(t)-24e^{-2t}u(t)$.

Solution

a) 4, note that $t=5$ does not belong to the integration interval.

b) 20

c) -4

d) $y(0^+)=3$

e) $z(t)=12e^{-2t}u(t)$

Question 2 (8 points)

The action of an LTI system with an input signal $x(t)$ and output signal $y(t)$ is described by the differential equation

$$y''(t)+4y'(t)+13y(t)=x(t) \quad \text{for } t>0^-$$

with initial conditions $y(0^-)=4$ and $y'(0^-)=4$.

a) Determine the impulse response $h(t)$ of the system.

The zero-input response $y_{\mathrm{zi}}(t)$ of the system can be written as

$$y_{\mathrm{zi}}(t)=Ae^{-\alpha t}\cos(\omega t+\varphi)u(t).$$

b) Determine A, α, ω, and φ.

Solution

a) To determine the impulse response, take $x(t) = \delta(t)$ and set the initial conditions to zero. A one-sided Laplace transform gives the transfer function

$$H(s) = \frac{1}{s^2 + 4s + 13} = \frac{1}{3} \frac{3}{(s+2)^2 + 3^2}.$$

The inverse Laplace transform of this transfer function is

$$h(t) = \frac{1}{3} e^{-2t} \sin(3t) u(t).$$

b) Set the input signal to zero $x(t) = 0$, apply the one-sided Laplace transform to the differential equation, and take the initial conditions into account to obtain

$$Y_{\mathrm{zi}}(s) = \frac{4s + 20}{s^2 + 4s + 13} = \frac{C}{s + 2 - 3\mathrm{j}} + \frac{C^*}{s + 2 + 3\mathrm{j}}$$

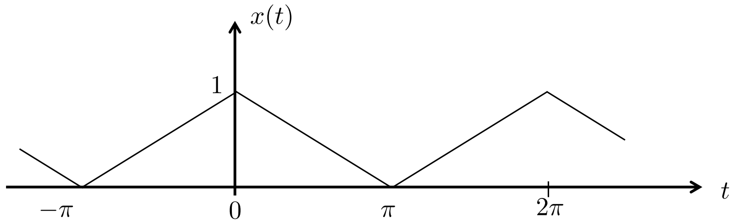
with $C = 2(1 - \mathrm{j}) = 2\sqrt{2}e^{-\mathrm{j}\pi/4}$. An inverse Laplace transform now gives

$$\begin{aligned} y_{\mathrm{zi}}(t) &= 2\sqrt{2}e^{-2t} \left[e^{\mathrm{j}(3t - \pi/4)} + e^{-\mathrm{j}(3t - \pi/4)} \right] u(t) \\ &= 4\sqrt{2}e^{-2t} \cos(3t - \pi/4) u(t). \end{aligned}$$

We find that $A = 4\sqrt{2}$, $\alpha = 2$, $\omega = 3$, and $\varphi = -\pi/4$.

Question 3 (10 points)

Consider the real-valued continuous-time periodic signal $x(t)$ depicted below:



The Fourier series expansion of $x(t)$ is given by

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}, \quad t \in [t_0, t_0 + T_0), \qquad X_k = \frac{1}{T_0} \int_{t_0}^{t_0 + T_0} x(t) e^{-jk\Omega_0 t} dt, \quad k \in \mathbb{Z}. \tag{1}$$

a) Without explicitly computing the Fourier series, what can you say about the spectrum of $x(t)$: Is it discrete or continuous, is it real, imaginary of complex-valued, and do we have symmetry in the spectrum? Motivate your answer.

b) Compute the Fourier series expansion of $x(t)$ using (1).

c) Compute the Fourier coefficients using the Laplace transform.

d) Give a sketch of the magnitude and phase spectrum of $x(t)$.

Assume the signal $x(t)$ is the input of an LTI system of which the output is $y(t) = \pi \dfrac{dx(t)}{dt}$.

e) Compute the Fourier series expansion of $y(t)$.

f) What is the order of decay of both spectra? Explain which one decays fastest and why.

Solution

a) Since x is a periodic signals, the spectrum is a line spectrum. Moreover, since x is even symmetric, the spectrum is real and since x is real valued, the spectrum is conjugate symmetric. That is, the magnitude spectrum is even symmetric and the phase spectrum is odd symmetric.

b) Since x is even symmetric, we have ($\Omega_0 = 1$)

$$\int_{-\pi}^{\pi} x(t)e^{-jkt}dt = \int_{-\pi}^{\pi} x(t)(\cos(kt) - j\sin(kt))dt = \int_{-\pi}^{\pi} x(t)\cos(kt)dt.$$

Hence,

$$X_k = \frac{1}{\pi}\int_0^{\pi}(1 - \frac{t}{\pi})\cos(kt)dt = \underbrace{\frac{1}{\pi}\int_0^{\pi}\cos(kt)dt}_{I} - \underbrace{\frac{1}{\pi}\int_0^{\pi}\frac{t}{\pi}\cos(kt))dt}_{II}.$$

$k \neq 0$:

I :
$$\frac{1}{\pi}\int_0^{\pi}\cos(kt)dt = -\frac{1}{\pi k}\sin(kt)\Big|_0^{\pi} = 0.$$

II :
$$-\frac{1}{\pi^2}\int_0^{\pi} t\cos(kt)dt = \frac{1}{\pi^2 k}t\sin(kt)\Big|_0^{\pi} - \frac{1}{\pi^2 k}\int_0^{\pi}\sin(kt)dt$$

$$= 0 - \frac{1}{(\pi k)^2}\cos(kt)\Big|_0^{\pi} = \frac{1}{(\pi k)^2}(\Big(1 - (-1)^k\Big).$$

$k = 0$:

I :
$$\frac{1}{\pi}\int_0^{\pi} dt = 1.$$

II :
$$-\frac{1}{\pi^2}\int_0^{\pi} tdt = -\frac{1}{2\pi^2}t^2\Big|_0^{\pi} = -\frac{1}{2},$$

so that $X_0 = \frac{1}{2}$. Note that X_k is indeed real.

c) We have that $x(t) = \frac{1}{\pi}r(t+\pi) - \frac{2}{\pi}r(t) + \frac{1}{\pi}r(t-\pi)$. Applying the Laplace transform yields

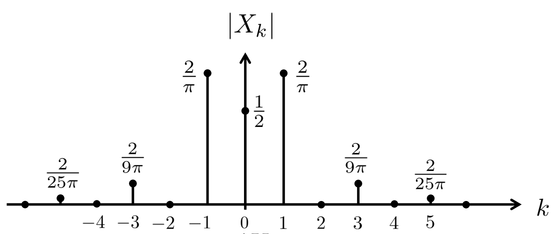
$$X(s) = \frac{1}{\pi s^2}\left(e^{s\pi} + e^{-s\pi} - 2\right).$$

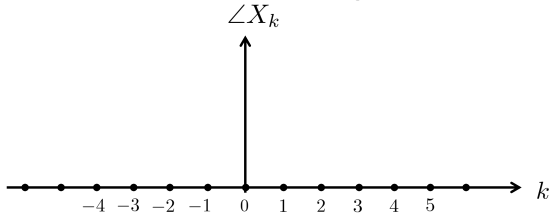
so that the Fourier coefficients are given by ($k \neq 0$),

$$X_k = \frac{1}{2\pi}X(s)\Big|_{s=jk} = \frac{-1}{2(\pi k)^2}\left(e^{jk\pi} + e^{-jk\pi} - 2\right) = \frac{1}{(\pi k)^2}(\Big(1 - (-1)^k\Big).$$

For $k = 0$ we note that $\lim_{s\to 0} X(s) = \pi$, so that $X_0 = \frac{1}{2}$.

d) The magnitude and phase spectrum $|X_k|$ and $\angle X_k$ are:





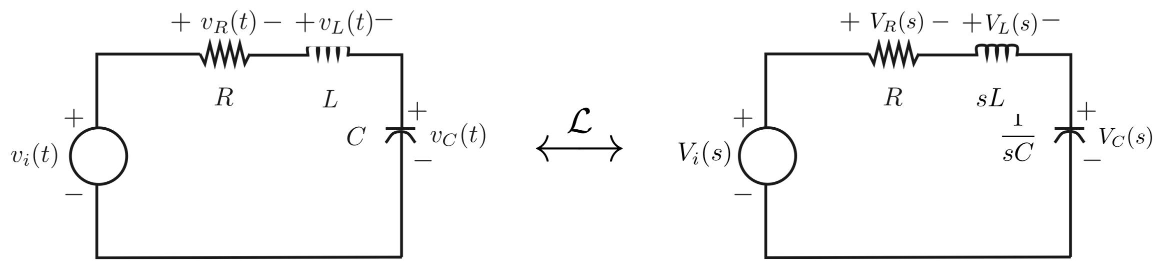
e) Since $y(t) = \pi \frac{dx(t)}{dt}$, we have

$$Y_k = jk\pi X_k = \begin{cases} 0, & k \text{ even}, \\ \frac{2j}{\pi k}, & k \text{ odd}. \end{cases}$$

f) The decay of X_k is $\mathcal{O}(1/k^2)$ since x has is continuous but not continuously differentiable. The decay of Y_k is slower ($\mathcal{O}(1/k)$) since y has discontinuities (square wave).

Question 4 (10 points)

Consider the RLC circuit depicted below:



Let $H_L(s)$ denote the the transfer function of the (input) voltage source to the (output) voltage across the inductor,

$$H_L(s) = \frac{V_L(s)}{V_i(s)}, \quad s \in \text{ROC}.$$

a) Give an expression for $H_L(s)$ in terms of R, L and C.

For simplicity, now assume that $R = L = C = 1$.

b) Determine the poles and zeros of the system and draw them in the complex s-plane.

Is the system BIBO stable? Motivate your answer.

c) Give an expression for the frequency, magnitude and phase response of the system.

d) Sketch the magnitude and phase response and indicate the values of $|H_L(j\Omega)|$ and $\angle H_L(j\Omega)$ for the (angular) frequencies $\Omega = 0$ and $\Omega = \pm\infty$.

e) Suppose we consider the transfer to the output across the resistor or capacitor, given by $H_R(s)$ and $H_C(s)$, respectively. Argue whether the corresponding frequency responses are low-, band- or highpass. Motivate your answer by considering the pole-zero locations of the transfer functions, and indicate how they differ from the pole-zero locations of $H_L(s)$.

Solution

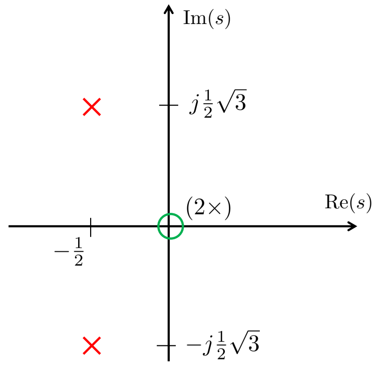
a) We have

$$H_L(s) = \frac{s^2LC}{s^2LC + sRC + 1}.$$

b)

$$H(s) = \frac{s^2}{s^2+s+1} = \frac{s^2}{(s+\frac{1}{2}(1-j\sqrt{3}))(s+\frac{1}{2}(1+j\sqrt{3}))}.$$

Hence, we have two zero at $s = 0$ and two poles at $s = -\frac{1}{2} \pm j\frac{1}{2}\sqrt{3}$. The causal system is BIBO stable since all poles lie in the left-hand plane.



b) The frequency response is given by

$$H(\Omega) = H(s)\Big|_{s=j\Omega} = \frac{-\Omega^2}{1-\Omega^2+j\Omega}.$$

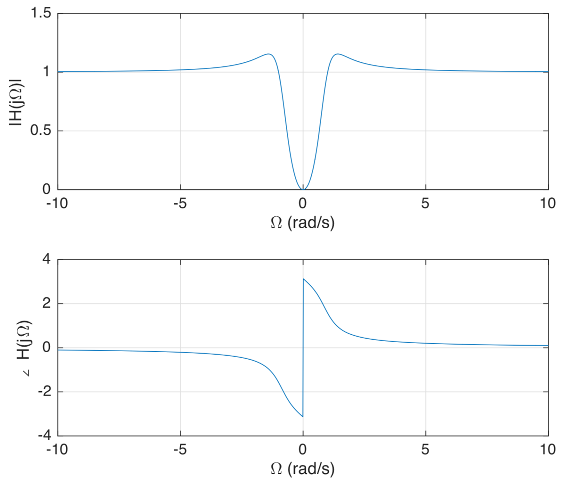
Hence, the magnitude response is given by

$$|H(\Omega)| = \frac{\Omega^2}{\sqrt{(1-\Omega^2)^2+\Omega^2}}$$

and the phase response by

$$\angle H(\Omega) = \pi - \tan^{-1}\left(\frac{\Omega}{1-\Omega^2}\right).$$

c) We have $|H(0)| = 0$ and $|H(\pm\infty)| = 1$, Since we have two zeros at $s = 0$, we get a jump of 2π radians at $\Omega = 0$. Hence, $\angle H(0^+) = \pi$ and $\angle H(0^-) = -\pi$. Moreover, $\angle H(\pm\infty) = 0$.



d) The transfer function H_R and H_C are given by

$$H_R(s) = \frac{s}{s^2+s+1}, \quad H_C(s) = \frac{1}{s^2+s+1}.$$

They differ from H_L is the number of zeros (at $s = 0$). Hence, removing one zero from $H_L(s)$ will result in a band-pass filter, since there is still one zero left at $s = 0$ but $|H_R(\pm\infty| = 0$. If we remove both zeros, we obtain a low-pass filter since in that case $|H(0)| = 1$ and certainly $|H_R(\pm\infty| = 0$.