Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

EE2S11 SIGNALS AND SYSTEMS

Resit exam, 11 July 2022, 13:30–16:30

Closed book; one A4 (two sides) of handwritten notes permitted. No other tools except a basic pocket calculator permitted.

This exam consists of seven questions (37 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each answer sheet.

Question 1 (6 points)

The behavior of a SISO LTI system is governed by the differential equation

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 2y(t) = x(t) + \frac{\mathrm{d}x(t)}{\mathrm{d}t},$$

where $x(t)$ is the input signal and $y(t)$ is the output signal.

(a) Determine the transfer function $H(s)$ and the impulse response $h(t)$ of the system.

(b) Determine the output signal $y(t)$ in case the input signal is given by

$$x(t) = (1-t)e^{-t}u(t)$$

and the system is initially at rest.

(c) Determine $\lim_{t\to\infty} y(t)$ in case the input signal is given by

$$x(t) = 2u(t)$$

and the system is initially at rest.

Solution

2p (a) A one-sided Laplace transform gives

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+1}{s^2+2s+2} = \frac{s+1}{(s+1)^2+1}.$$

Using a table of Laplace transforms, we find

$$h(t) = e^{-t}\cos(t)u(t).$$

3p (b) The one-sided Laplace transform of $x(t)$ is given by

$$X(s) = \frac{1}{s+1} - \frac{1}{(s+1)^2} = \frac{s}{(s+1)^2},$$

and the transformed output signal becomes

$$Y(s) = H(s)X(s) = \frac{s}{[(s+1)^2+1](s+1)} = \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1} - \frac{1}{s+1}.$$

An inverse Laplace transform gives

$$y(t) = e^{-t}[\cos(t) + \sin(t) - 1]u(t).$$

1p (c) $\lim_{t\to\infty} y(t) = 1$. Directly follows from the differential equation or using

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s).$$

Question 2 (6 points)

(a) The one-sided Laplace transform of a causal signal $x(t)$ is given by

$$X(s) = \frac{3s+9}{s^2+2s+10}, \qquad \text{Re}(s) > -1.$$

Determine $x(t)$.

(b) Determine the inverse Laplace transform of the signal

$$Y(s) = \frac{2}{s^2(s-1)}, \quad \text{Re}(s) > 1.$$

Solution

(a)

$$X(s) = \frac{3(s+3)}{(s+1)^2+9} = 3\frac{s+1}{(s+1)^2+9} + 2\frac{3}{(s+1)^2+9}, \quad \text{Re}(s) > -1.$$

An inverse Laplace transform now gives $x(t) = e^{-t}[3\cos(3t) + 2\sin(3t)]u(t)$.

(b) We have

$$Y(s) = -2\frac{1}{s^2} - 2\frac{1}{s} + 2\frac{1}{s-1}, \quad \text{Re}(s) > 1.$$

An inverse Laplace transform gives $y(t) = 2(e^t - t - 1)u(t)$.

Question 3 (5 points)

Given the periodic signal $x(t)$ with fundamental period $T_0 = \pi$ and

$$x(t) = \cos(t), \quad 0 < t < \pi\,.$$

(a) Determine the average value of this signal.

(b) Expand $x(t)$ in a Fourier sine series.

(c) The Fourier coefficients of $x(t)$ decay as $1/k$ for $k \to \infty$. Explain why.

Solution

1p (a) Average is $X_0 = 0$.

3p (b)

$$x(t) = \frac{8}{\pi}\sum_{k=1}^{\infty} \frac{k}{4k^2-1}\sin(2kt)$$

1p (c) $x(t)$ is discontinuous at the end points.

Question 4 (8 points)

(a) Given the signal $x[n] = [\cdots, 0, \boxed{1}, 2, 3, 0, 0, \cdots]$, where the 'box' denotes the value for $n = 0$. Determine $r[n] = x[n] * x[-n]$ using the convolution sum.

(b) Given an input signal $x[n] = \left(\frac{1}{3}\right)^n u[n] + 2^n u[-n]$, where $u[n]$ is the unit step function. Determine the z-transform $X(z)$, also specify the ROC.

(c) Let

$$X(z) = \frac{2z^2}{2z^2 + z - 1}, \qquad \text{ROC: } |z| > 1\,.$$

Compute the inverse z-transform $x[n]$.

(d) Let $h[n] = [\cdots, 0, \boxed{1}, 2, 0, -2, -1, 0, \cdots]$.

Compute the DTFT $H(e^{j\omega})$ and subsequently determine the magnitude response $|H(e^{j\omega})|$.

(e) Compute all poles and zeros of $X(z)$ in (c) and draw a pole-zero plot.

Solution

2p (a) Let $y[n] = x[-n] = [\cdots, 0, 3, 2, \boxed{1}, 0, 0, \cdots]$, then $r[n] = \sum_{k=1}^{2} x[k]y[n-k]$,

$k=0:$	$1y[n]:$	3	2	$\boxed{1}$	0	0	$0\cdots$
$k=1:$	$2y[n-1]:$	0	6	$\boxed{4}$	2	0	$0\cdots$
$k=2:$	$3y[n-2]:$	0	0	$\boxed{9}$	6	3	$0\cdots$
	$r[n]:$	3	8	$\boxed{14}$	8	3	$0\cdots$

1.5p (b)

$$X(z) = \frac{1}{1-(\frac{1}{3})z^{-1}} + \frac{1}{1-\frac{1}{2}z} = \frac{z}{z-\frac{1}{3}} + \frac{1}{1-\frac{1}{2}z} = \frac{-\frac{1}{2}z^2 + 2z^{-1} - \frac{1}{3}}{-\frac{1}{2}z^2 - \frac{7}{6}z - \frac{1}{3}}$$

ROC: $\frac{1}{3} < |z| < 2$.

2p (c) The pole locations are $z = -1$ and $z = \frac{1}{2}$. Looking at the ROC, we expect a causal impulse response.

$$\begin{aligned} X(z) &= \frac{1}{1 + \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2}} \\ &= \frac{\frac{2}{3}}{1 + z^{-1}} + \frac{\frac{1}{3}}{1 - \frac{1}{2}z^{-1}} \end{aligned}$$

Then $x[n] = \frac{2}{3}(-1)^n u[n] + \frac{1}{3}\left(\frac{1}{2}\right)^n u[n]$.

1.5p (d)

$$H(z) = 1 + 2z^{-1} - 2z^{-3} - z^{-4}$$

$$\begin{aligned} H(e^{j\omega}) &= 1 + 2e^{-j\omega} - 2e^{-j3\omega} - e^{-j4\omega} \\ &= e^{-j2\omega}\left(e^{j2\omega} - e^{-j2\omega} + 2e^{j\omega} - 2e^{-j\omega}\right) \\ &= e^{-j2\omega}\left(2j\sin(2\omega) + 4j\sin(\omega)\right) \end{aligned}$$

Then

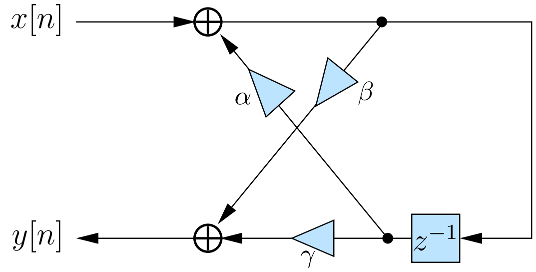
$$|H(e^{j\omega})| = \left|2\sin(2\omega) + 4\sin(\omega)\right|$$

1p (e) Poles: $z = -1$ and $z = \frac{1}{2}$.

Zeros: $z = 0$ (two times)

Question 5 (4 points)

Consider this realization of a causal system:



(a) Determine the transfer function $H(z)$.

(b) Under which conditions on the coefficients (α, β, γ) is this a stable system?

(c) Draw the "direct form no. II" realization, also specify the coefficients.

Solution

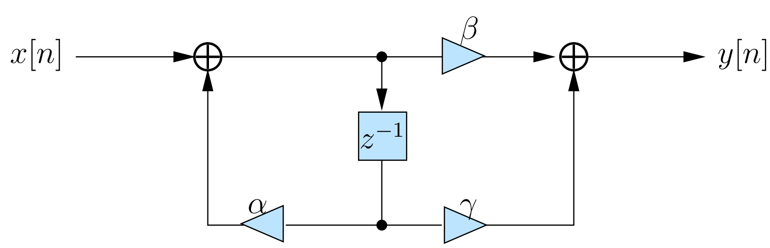
2p (a) Denote the input of the delay element by the signal $P(z)$; the output of it is $P(z)z^{-1}$.

$$\begin{cases} Y(z) &= \beta X(z) + P(z)z^{-1}(\alpha\beta + \gamma) \\ P(z) &= X(z) + \alpha P(z)z^{-1} \end{cases}$$

$$\Rightarrow \begin{cases} P(z) &= \frac{X(z)}{1-\alpha z^{-1}} \\ H(z) &= \beta + \frac{(\alpha\beta+\gamma)z^{-1}}{1-\alpha z^{-1}} = \frac{\beta+\gamma z^{-1}}{1-\alpha z^{-1}} \end{cases}$$

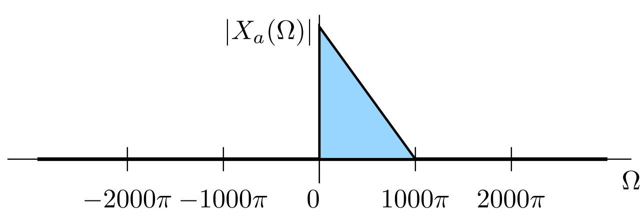
1p (b) $|\alpha| < 1$.

1p (c)



Question 6 (3 points)

A complex continuous-time signal $x_a(t)$ has amplitude spectrum as shown in the figure below. The signal is sampled with period $T = 2$ ms; the resulting signal is $x[n] = x_a(nT)$.

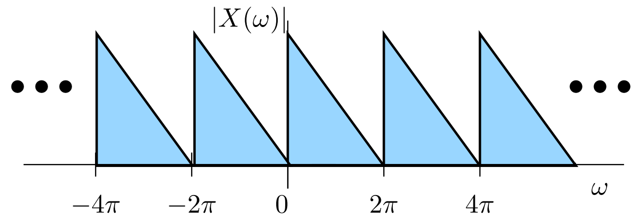


(a) Draw the amplitude spectrum $|X(\omega)|$ of $x[n]$. Also indicate values on the axes.

(b) Can we perfectly recover $x_a(t)$ from $x[n]$? (Why, or why not?)

Solution

2p (a) $\Omega_s = 2\pi/T = 1000\pi$. Note that Ω_s will map to $\omega_s = 2\pi$. The spectrum will become periodic with period Ω_s.



1p (b) Note that there is no destructive aliasing. We can do an ideal reconstruction (the result will be complex), followed by a filter that passes only the *positive* frequencies between 0 and 1000π.

Question 7 (5 points)

Consider an analog integrator with transfer function $H_a(s) = \dfrac{1}{s}$. We aim to implement this in the digital domain using the bilinear transformation.

(a) What is the resulting transfer function $H(z)$ if we apply the bilinear transformation?

(b) What is the corresponding impulse response $h[n]$? Is this indeed an integrator?

(c) Compute and draw the amplitude and phase response of both $H_a(s)$ and $H(z)$. Where do (don't) they match?

(d) More in general: what are two advantages of the filter design technique using the bilinear transformation?

Solution

1p (a)

$$H(s) = \frac{1}{s}; \qquad \text{substitute } s = \frac{1 - z^{-1}}{1 + z^{-1}};$$

$$H(z) = \frac{1 + z^{-1}}{1 - z^{-1}}$$

1p (b) The impulse response is $h[n] = u[n] + u[n-1]$.

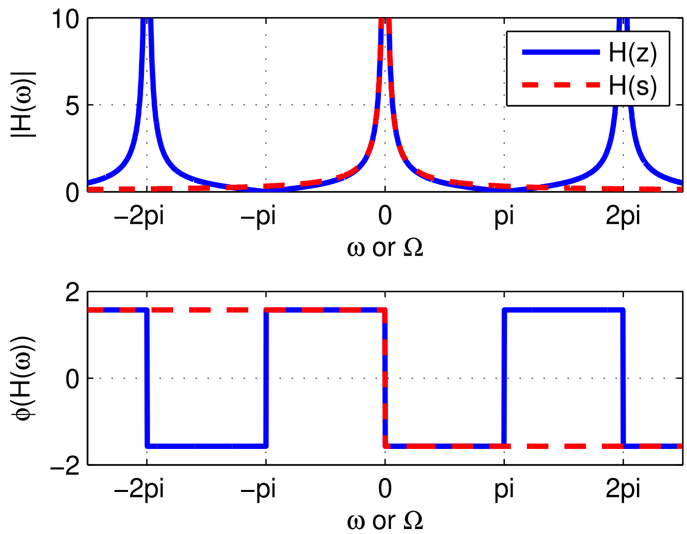
This is similar to a digital integrator $u[n]$, except for a factor 2 and the sample $h[0]$.

2p (c)

$$H(s) = \frac{1}{s} \quad \Rightarrow \quad H(j\Omega) = -j\frac{1}{\Omega} \quad \Rightarrow \quad |H(j\Omega)| = \frac{1}{\Omega}; \quad \phi(H(j\Omega) = \begin{cases} \pi/2, & \Omega < 0, \\ -\pi/2, & \Omega > 0 \end{cases}$$

$$H(z) = \frac{1 + z^{-1}}{1 - z^{-1}} \quad \Rightarrow \quad H(e^{j\omega}) = \frac{1 + e^{-j\omega}}{1 - e^{-j\omega}} = -j\frac{\cos(\omega/2)}{\sin(\omega/2)}$$

$$|H(\omega)| = |1/\tan(\omega/2)|; \quad \phi(H(\omega)) = \begin{cases} \pi/2, & -\pi < \omega < 0, \\ -\pi/2, & 0 < \omega < \pi \end{cases}$$



On the interval $[-\pi, \pi]$, the responses are nearly identical; outside this interval, $H(\omega)$ is periodic but $H(\Omega)$ is not.

1p (d) Stability is maintained; no aliasing due to the transformation: the entire left half plane transforms one-to-one to the entire unit disc.

We can use standard analog-domain design techniques (including high-pass filter design); the filter order is maintained.