Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

# Resit exam EE2S11 SIGNAL PROCESSING
# July 21, 2020
# Block 1 (13:30-15:00)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted.
Upload answers during 14:55–15:10

This block consists of three questions (30 points); more than usual, and this will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.

### Question 1 (10 points)

Given a SISO system with input signal $x(t)$ and output signal $y(t)$. For $T_1 \geq 0$ and $T_2 \geq 0$ and $T_1 + T_2 \neq 0$, the output signal $y(t)$ is related to the input signal $x(t)$ by

$$y(t) = \frac{1}{T_1 + T_2} \int_{\tau=t-T_1}^{t+T_2} x(\tau)\,\mathrm{d}\tau.$$

(a) The system is called a sliding window averager. Explain why.

(b) Is this system linear? Motivate your answer.

(c) Is this system time-invariant? Motivate your answer.

(d) Determine the transfer function of the system. What is its ROC?

(e) Determine the impulse response of the system.

(f) Is the system causal for $T_1 > 0$ and $T_2 > 0$? Motivate your answer.

(g) Is the system causal for $T_1 > 0$ and $T_2 = 0$? Motivate your answer.

### Solution

(a) For each time instant $t$ the output is the arithmetic average of the input signal taken over the interval $(t - T_1, t + T_2)$.

(b) Let $y_i(t)$ denote the output signals that correspond to the input signals $x_i(t)$, $i = 1, 2$. Given the input signal $x(t) = \alpha x_1(t) + \beta x_2(t)$, where $\alpha$ and $\beta$ are constants, we have

$$\begin{aligned} y(t) &= \frac{1}{T_1 + T_2} \int_{\tau=t-T_1}^{t+T_2} x(\tau)\,\mathrm{d}\tau = \frac{1}{T_1 + T_2} \int_{\tau=t-T_1}^{t+T_2} [\alpha x_1(t) + \beta x_2(t)]\,\mathrm{d}\tau \\ &= \alpha y_1(t) + \beta y_2(t). \end{aligned}$$

Linear combination of input signals leads to the same linear combination of the corresponding output signals. System is linear.

(c) Let $w(t)$ be the output signal of the system that corresponds to the input signal $v(t)$. In other words, we have

$$w(t) = \frac{1}{T_1 + T_2} \int_{\tau=t-T_1}^{t+T_2} v(\tau)\,\mathrm{d}\tau.$$

Now let $x(t) = v(t-a)$ be a time-shifted version of the input signal with time shift $a$. The output signal $y(t)$ that corresponds to this input signal is

$$\begin{aligned} y(t) &= \frac{1}{T_1 + T_2} \int_{\tau=t-T_1}^{t+T_2} x(\tau)\,\mathrm{d}\tau = \frac{1}{T_1 + T_2} \int_{\tau=t-T_1}^{t+T_2} v(\tau - a)\,\mathrm{d}\tau \\ &\overset{p=\tau-a}{=} \frac{1}{T_1 + T_2} \int_{p=t-a-T_1}^{t-a+T_2} v(p)\,\mathrm{d}p = w(t-a). \end{aligned}$$

A time shift in the input leads to a time-shifted output with the same time shift. System is time-invariant.

(d) System is LTI so we know that for an input signal $x(t) = e^{st}$ the output signal will be $y(t) = H(s)e^{st}$, where $H(s)$ is the transfer function. Substitution gives

$$y(t) = \frac{1}{T_1 + T_2} \int_{\tau=t-T_1}^{t+T_2} x(\tau)\,\mathrm{d}\tau = \frac{1}{T_1 + T_2} \int_{\tau=t-T_1}^{t+T_2} e^{s\tau}\,\mathrm{d}\tau = \frac{1}{s(T_1 + T_2)} \left(e^{sT_2} - e^{-sT_1}\right) e^{st}$$

and we observe that

$$H(s) = \frac{1}{T_1 + T_2} \left( \frac{e^{sT_2}}{s} - \frac{e^{-sT_1}}{s} \right).$$

The ROC $= \mathbb{C}$, there is no pole at $s = 0$.

(e) Inverse Laplace transform gives

$$h(t) = \frac{1}{T_1 + T_2} \left[ u(t + T_2) - u(t - T_1) \right].$$

Can also be seen directly from the given input-output relation, of course.

(f) No. $h(t) \neq 0$ for $t < 0$. Can also be seen from the given input-output relation, of course.

(g) Yes. In this case $h(t) = 0$ for $t < 0$. Can also be seen from the input-output relation.

**Question 2 (10 points)**

(a) Determine the Laplace transform $F(s)$ of the signal

$$f(t) = \sinh(t)u(t),$$

where $u(t)$ is the Heaviside unit step function.

(b) What is the ROC of $F(s)$?

For $t > 0$, the behavior of a system with input signal $x(t)$ and output signal $y(t)$ is governed by the differential equation

$$\frac{\mathrm{d}^4 y}{\mathrm{d}t^4} - y = x(t).$$

At $t = 0$, $y$ and its first three derivatives vanish.

(c) Determine the impulse response $h(t)$ of the system.

(d) True or false: the output signal $y(t)$ of the system for a given input signal $x(t)$ and with vanishing initial conditions is given by

$$y(t) = \frac{1}{2}\int_{\tau=0}^{t} [\sinh(t-\tau) - \sin(t-\tau)]\, x(\tau)\, \mathrm{d}\tau, \quad t > 0.$$

Motivate your answer.

**Solution**

(a) We have

$$\begin{aligned} F(s) &= \int_{t=0}^{\infty} \sinh(t) e^{-st}\, \mathrm{d}t = \int_{t=0}^{\infty} \frac{e^{t} - e^{-t}}{2} e^{-st}\, \mathrm{d}t \\ &= \frac{1}{2}\int_{t=0}^{\infty} e^{-(s-1)t} - e^{-(s+1)t}\, \mathrm{d}t = \frac{1}{2}\lim_{T\to\infty}\int_{t=0}^{T} e^{-(s-1)t}\, \mathrm{d}t - \frac{1}{2}\lim_{T\to\infty}\int_{t=0}^{T} e^{-(s+1)t}\, \mathrm{d}t \end{aligned}$$

For the first integral, we have

$$\frac{1}{2}\lim_{T\to\infty}\int_{t=0}^{T} e^{-(s-1)t}\, \mathrm{d}t = \frac{1}{2}\frac{1}{s-1} \quad \text{for } \mathrm{Re}(s) > 1.$$

For the second integral we have

$$\frac{1}{2}\lim_{T\to\infty}\int_{t=0}^{T} e^{-(s+1)t}\, \mathrm{d}t = \frac{1}{2}\frac{1}{s+1} \quad \text{for } \mathrm{Re}(s) > -1.$$

Consequently,

$$F(s) = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) = \frac{1}{s^2-1} \quad \text{for } \mathrm{Re}(s) > 1$$

(b) ROC $= \{s \in \mathbb{C}; \mathrm{Re}(s) > 1\}$.

(c) Impulse response $h(t)$ is the response of the system to a delta input only (initial conditions vanish). In other words, $h$ satisfies

$$\frac{\mathrm{d}^4 h}{\mathrm{d}t^4} - h = \delta(t)$$

with vanishing initial conditions. Applying the one-sided Laplace transform to this equation and taking the initial conditions into account, we obtain

$$(s^4-1)H(s) = 1 \quad \text{or} \quad H(s) = \frac{1}{s^4-1} = \frac{1}{(s^2-1)(s^2+1)} = \frac{1}{2}\left(\frac{1}{s^2-1} - \frac{1}{s^2+1}\right)$$

for $\mathrm{Re}(s) > 0$. An inverse Laplace transform now gives

$$h(t) = \frac{1}{2}[\sinh(t) - \sin(t)]\, u(t).$$

(d) Output signal for an arbitrary input signal is

$$y(t) = \int_{\tau=0}^{\infty} h(t-\tau)x(\tau)\, \mathrm{d}\tau = \frac{1}{2}\int_{\tau=0}^{t} [\sinh(t-\tau) - \sin(t-\tau)]\, x(\tau)\, \mathrm{d}\tau.$$

Statement is true.

**Question 3 (10 points)**

Let $x(t)$ be a periodic signal with fundamental period $T_0 = 4$. On the interval $(-2, 2)$, $x(t)$ is given by

$$x(t) = t^2, \quad t \in (-2, 2).$$

(a) What can you say about the decay of the Fourier coefficients as $|k| \to \infty$ without computing these coefficients explicitly?

(b) Determine $X_0$, the dc-component of the signal $x(t)$.

(c) Determine the Fourier coefficients $X_k$ for $k \neq 0$.

(d) Determine the power $P_x$ of the signal.

(e) Use Parseval's power relation to show that

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$

**Solution**

(a) $x(t)$ is continuous, but its first derivative is not. Coefficients decay as $1/k^2$ as $|k| \to \infty$.

(b)

$$X_0 = \frac{1}{4} \int_{t=-2}^{2} t^2 \, \mathrm{d}t = \frac{1}{2} \int_{t=0}^{2} t^2 \, \mathrm{d}t = \frac{4}{3}$$

(c) For $k \neq 0$

$$X_k = \frac{1}{4} \int_{t=-2}^{2} t^2 \cos(k\Omega_0 t) \, \mathrm{d}t = \frac{1}{2} \int_{t=0}^{2} t^2 \cos(k\Omega_0 t) \, \mathrm{d}t = \frac{8}{\pi^2 k^2}(-1)^k.$$

(d)

$$P_x = \frac{1}{4} \int_{t=-2}^{2} t^4 \, \mathrm{d}t = \frac{1}{2} \int_{t=0}^{2} t^4 \, \mathrm{d}t = \frac{16}{5}.$$

(e) Parseval's power relation:

$$\begin{aligned} P_x &= \sum_{k=-\infty}^{\infty} |X_k|^2 = |X_0|^2 + \sum_{k=-\infty, k\neq 0}^{\infty} |X_k|^2 \\ &= \frac{16}{9} + 2\sum_{k=1}^{\infty} \frac{64}{\pi^4 k^4} \\ &= \frac{16}{9} + \frac{128}{\pi^4} \sum_{k=1}^{\infty} \frac{1}{k^4} \end{aligned}$$

from which it follows that

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \left( \frac{16}{5} - \frac{16}{9} \right) \cdot \frac{\pi^4}{128} = \frac{\pi^4}{90}$$

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# Resit exam EE2S11 SIGNAL PROCESSING
# July 21, 2020
# Block 2 (15:00-16:30)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted. Upload answers during 16:30–16:45

This block consists of four questions (27 points); more than usual, and this will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.

**Question 4 (10 points)**

(a) Given the signals $x[n] = [\cdots, 0, 1, \boxed{2}, 3, 0, \cdots]$ and $h[n] = [\cdots, \boxed{0}, 1, 2, 0, \cdots]$. Determine $y[n] = h[n] * x[n]$ using the convolution sum.

(b) Given an input signal $x[n] = \left(\frac{1}{4}\right)^n u[n]$, and a system described by the difference equation

$$y[n] = 2x[n] - \frac{1}{2}y[n-1]\,.$$

Determine the output signal $y[n]$.

(c) Consider

$$X(z) = \frac{z^2 - 1}{z^2 + 4}\,.$$

Make a pole-zero plot, and compute $x[n]$ for two cases: $(i)$ ROC: $|z| < 2$, and $(ii)$ ROC: $|z| > 2$.

(d) Given $x[n] = 2\,a^n \cos(\omega_0 n)$, with $|a| < 1$. Determine the DTFT $X(\omega)$.

**Solution**

(a) 1 pnt $y[n] = \sum_{k=1}^{2} h[k]x[n-k]$

| | | | | | |
|---|---|---|---|---|---|
| $k = 1 : x[n-1]$ : | $\boxed{1}$ | 2 | 3 | 0 | $\cdots$ |
| $k = 2 : 2x[n-2]$ : | $\boxed{0}$ | 2 | 4 | 6 | $0\cdots$ |
| $y[n]$ : | $\boxed{1}$ | 4 | 7 | 6 | $0\cdots$ |

(b) 3 pnt

$$X(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}\,, \qquad \text{ROC: } |z| > \frac{1}{4}$$

$$
\begin{aligned}
Y(z) &= \frac{2X(z)}{1+\frac{1}{2}z^{-1}} \\
&= \frac{2}{(1-\frac{1}{4}z^{-1})(1+\frac{1}{2}z^{-1})} \\
&= \frac{2/3}{1-\frac{1}{4}z^{-1}} + \frac{4/3}{1+\frac{1}{2}z^{-1}}\,. \\
y[n] &= \frac{2}{3}\left(\frac{1}{4}\right)^n u[n] + \frac{4}{3}\left(-\frac{1}{2}\right)^n u[n]\,.
\end{aligned}
$$

(c) 4 pnt Poles at $z = \pm 2j$, zeros at $z = \pm 1$.

$$
\begin{aligned}
X(z) &= \frac{z^2-1}{z^2+4} = \frac{1-z^{-2}}{1+4z^{-2}} \\
&= -\frac{1}{4} + \frac{5/4}{1+4z^{-2}} \\
&= -\frac{1}{4} + \frac{5/8}{1+2jz^{-1}} + \frac{5/8}{1-2jz^{-1}} \qquad = -\frac{1}{4} + \frac{j5/16\,z}{1-\frac{1}{2}jz} - \frac{j5/16\,z}{1+\frac{1}{2}jz}
\end{aligned}
$$

($i$) ROC $|z| < 2$: anticausal (but stable) response:

$$
\begin{aligned}
x[n] &= -\frac{1}{4}\delta[n] + \frac{5}{16}\left[j(\frac{1}{2}j)^{-n-1} - j(-\frac{1}{2}j)^{-n-1}\right] u[-n-1] \\
&= -\frac{1}{4}\delta[n] - \frac{5}{8}\left(\frac{1}{2}\right)^{-n-1} \sin\left(\frac{1}{2}\pi(-n-1)\right) u[-n-1]\,.
\end{aligned}
$$

($ii$) ROC $|z| > 2$: causal (but unstable) response:

$$
x[n] = -\frac{1}{4}\delta[n] + \frac{5}{8}\left[(2j)^n + (-2j)^n\right] u[n] = -\frac{1}{4}\delta[n] + \frac{5}{4}2^n \cos\left(\frac{1}{2}\pi n\right) u[n]\,.
$$

Many different (but equivalent) expressions are possible here. Check the solution using $\lim_{z\to\infty} X(z) = 1 = x[0]$.

(d) 2 pnt

$$
\begin{aligned}
a^n u[n] \quad &\rightarrow \quad \frac{1}{1-ae^{-j\omega}} \\
2\cos(\omega_0 n) = e^{j\omega_0 n} + e^{-j\omega_0 n} \quad &\rightarrow \quad 2\pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]\,.
\end{aligned}
$$

Using $x[n]\cdot y[n] \;\leftrightarrow\; \frac{1}{2\pi}X(\omega) * Y(\omega)$:

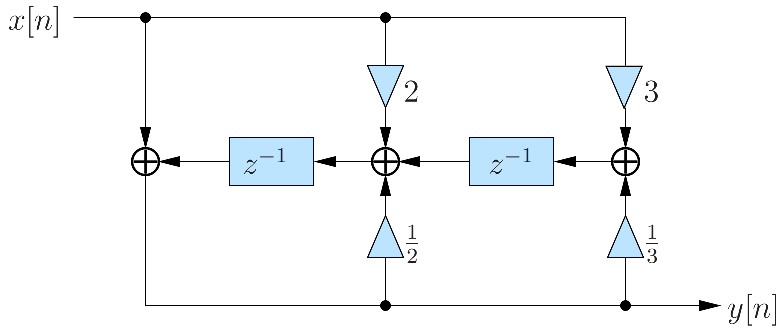
$$
X(\omega) = \frac{1}{1-ae^{-j(\omega-\omega_0)}} + \frac{1}{1-ae^{-j(\omega+\omega_0)}}\,.
$$

This expression could be rewritten as

$$
X(\omega) = \frac{2-2a\cos(\omega-\omega_0)}{1+a^2e^{-j2\omega}+2a\cos(\omega-\omega_0)}\,.
$$

**Question 5 (4 points)**

Consider the following system realization:




(a) Determine the transfer function $H(z)$.

(b) Is this a minimal realization? (Why?)

(c) Draw the corresponding Direct Form no. 2 realization.

**Solution**

(a) 2 pnt Call the inputs of the two delay elements $P(z)$ and $Q(z)$.

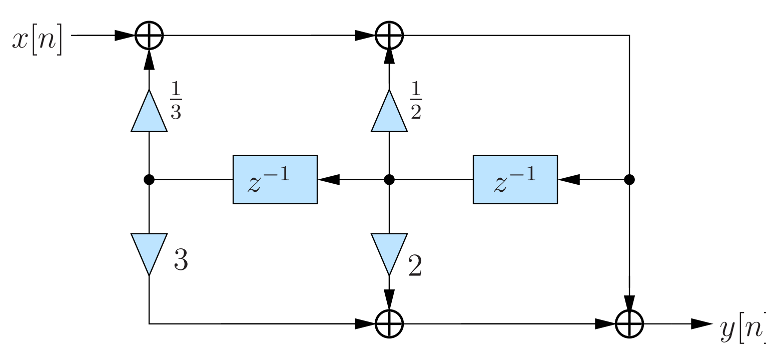
$$\begin{cases} P(z) &= 3X(z) + \frac{1}{3}Y(z) \\ Q(z) &= 2X(z) + \frac{1}{2}Y(z) + z^{-1}P(z) \\ Y(z) &= X(z) + z^{-1}Y(z) \end{cases}$$

$$Y(z) = X(z) + z^{-1}(2 + \frac{1}{2}Y(z)) + z^{-2}(3 + \frac{1}{3}X(z))$$

$$H(z) = \frac{1 + 2z^{-1} + 3z^{-3}}{1 - \frac{1}{2}z^{-1} - \frac{1}{3}z^{-2}}$$

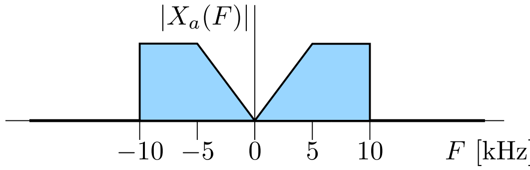
(b) 1 pnt Yes, $H(z)$ is 2nd order and the realization uses 2 delay elements.

(c) 1 pnt




**Question 6 (5 points)**

A continuous-time signal $x_a(t)$ has an amplitude spectrum $X_a(F)$ as shown below. The signal is sampled with period $T$ so that we obtain a series $x[n] = x_a(nT)$.




*For this question, draw the spectra at least for $\omega$ running from $-2\pi$ until $2\pi$.*

(a) What is the Nyquist frequency at which we would have to sample to avoid any aliasing?

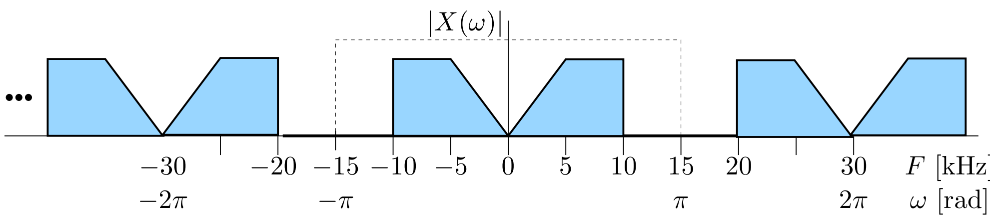
(b) We sample the signal at 30 kHz. Make a drawing of the resulting amplitude spectrum $|X(\omega)|$ of $x[n]$. Also mark the frequencies.

(c) After sampling, we apply an ideal digital highpass filter, with cutoff frequency $\omega_c = \frac{1}{3}\pi$. Make a drawing of the resulting amplitude spectrum $|Y(\omega)|$. Also mark the frequencies.

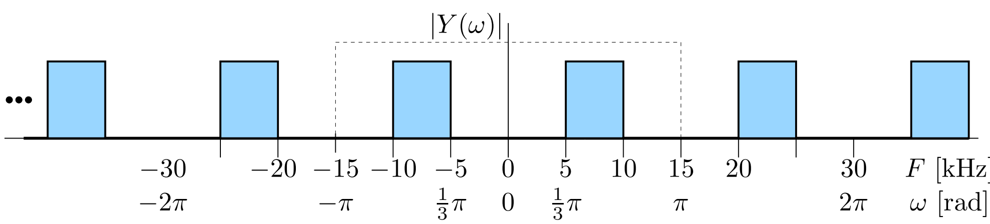
(d) After sampling, we invert every second sample of $x[n]$, resulting in $r[n] = (-1)^n x[n]$. Make a drawing of the resulting amplitude spectrum $|R(\omega)|$. Also mark the frequencies.

**Solution**

(a) 1 pnt 20 kHz.

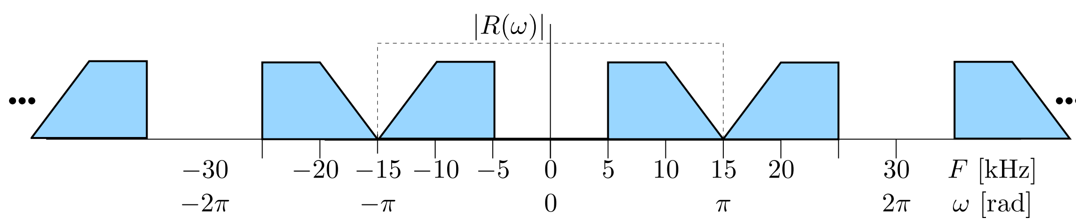



(b) 1 pnt




(c) 1 pnt

(d) 2 pnt The effect of this modulation by $e^{j\pi n}$ is a shift of the spectrum by $\pi$, i.e. $R(\omega) = X(\omega - \pi)$.



**Question 7 (8 points)**

In this question, we will design a Chebyshev type II lowpass filter $G(\Omega)$ with the following specifications:

| | |
|---|---|
| Third order | |
| Passband: | $F_p = 3$ kHz |
| Stopband: | $F_s = 5$ kHz |
| Minimal stopband damping: | 20 dB |

Recall that a template Chebyshev (type I) filter has amplitude response

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2 T_n^2(\Omega)}\,.$$

A Chebyshev type II filter $G(\Omega)$ is derived from type I in two steps. First,

$$|F(\Omega)|^2 = 1 - |H(\Omega)|^2 = \frac{\epsilon^2 T_n^2(\Omega)}{1+\epsilon^2 T_n^2(\Omega)}\,.$$

Next, apply a frequency transformation $\Omega \to \frac{\Omega_0}{\Omega}$:

$$|G(\Omega)|^2 = |F(\Omega_0/\Omega)|^2 = \frac{\epsilon^2 T_n^2(\Omega_0/\Omega)}{1+\epsilon^2 T_n^2(\Omega_0/\Omega)}\,.$$

(a) Recall that the third order Chebyshev polynomial is given by

$$T_3(\Omega) = 4\Omega^3 - 3\Omega\,.$$

Give a plot of $T_3(\Omega)$. Determine $\Omega$ for which $T_3(\Omega)$ is 0, 1, $\infty$.

(b) Draw plots for $|H(\Omega)|^2$, $|F(\Omega)|^2$ and $|G(\Omega)|^2$ (for $n = 3$ and $\Omega_0 = 1$).

Indicate values on the horizontal and vertical axes. Pay attention to accurately draw the ripples.

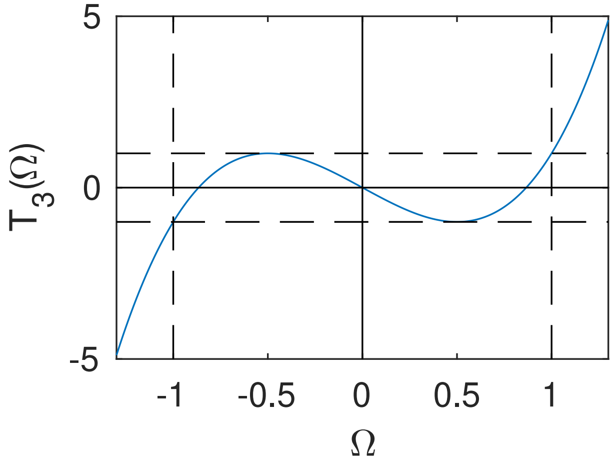
(c) Determine $\Omega_0$ and $\epsilon$ such that $G(\Omega)$ satisfies the specifications listed at the beginning of this question.

(d) How many dB is the maximal passband attenuation for this 3rd order Chebyshev II filter?

**Solution**

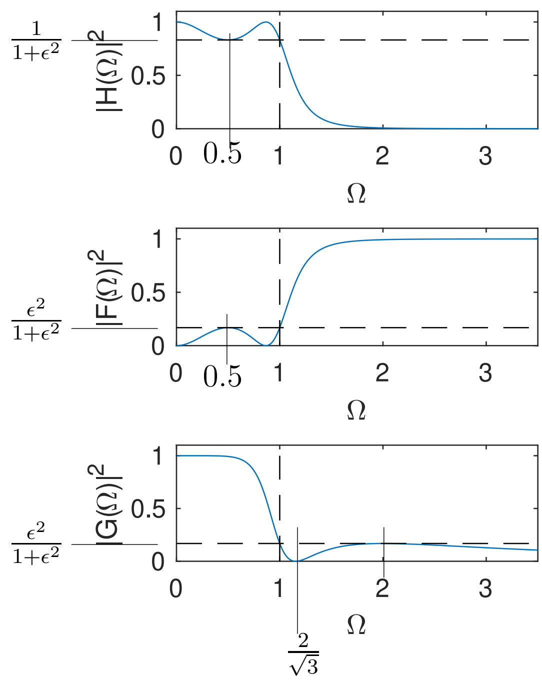
(a) 2 pnt

$$\begin{aligned}
T_3(\Omega) &= 0 \quad\Leftrightarrow\quad \Omega(4\Omega^2-3)=0 \quad\Rightarrow\quad \Omega=0 \text{ or } \Omega=\pm\frac{\sqrt{3}}{2}\\
T_3(\Omega) &= 1 \quad\Leftrightarrow\quad (\Omega-1)(4\Omega^2+4\Omega+1)=0 \quad\Rightarrow\quad \Omega=1 \text{ or } \Omega=-\frac{1}{2}\\
T_3(\Omega) &= \infty \quad\Leftrightarrow\quad \Omega=\infty
\end{aligned}$$




(b) 2 pnt Use the plot of $T_3(\Omega)$ to get the shape of the ripple right. E.g., $|H(\Omega)| = 1$, and there is one other point (at $\Omega = \sqrt{3}/2$) where $|H(\Omega)| = 1$.

The plot of $|F(\Omega)|$ is a transformation of the vertical axis and results in a highpass filter. The plot of $|G(\Omega)|$ is found after a transformation of the horizontal axis, $\Omega \to 1/\Omega$, which transforms a highpass into a lowpass.




(c) 2 pnt Take $\Omega_0 = 2\pi F_s = 10\pi \cdot 1000 = 31.4$ krad/s.

At $\Omega_0$, $T_3(\Omega_0/\Omega) = 1$, and

$$|G(\Omega_0)|^2 = \frac{\epsilon^2}{1+\epsilon^2} = 1 - \frac{1}{1+\epsilon^2} = 10^{-20/10} \quad \Leftrightarrow \quad \epsilon = \sqrt{\frac{1}{0.99} - 1} = 0.1005$$

(d) 2 pnt $\Omega_p = 2\pi \cdot 3000$, $T_3(\Omega_0/\Omega_p) = T_3(5/3) = 13.519$.

$$|G(\Omega_0)|^2 = 0.6486\,.$$

The maximal damping is $-10\log(0.6486) = 1.88$ dB.