Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

Exam EE2S11 Signals and Systems
Resit on complete course: 23 July 2019, 13:30–16:30

Closed book; one double-sided A4 page of handwritten notes permitted

This exam consists of seven questions (40 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.

Question 1 (5 points)

a) The one-sided Laplace transform of a causal signal $f(t)$ is given by

$$F(s) = \frac{5s+13}{s(s^2+4s+13)}, \qquad \mathrm{Re}(s) > 0.$$

Determine $f(t)$.

b) The one-sided Laplace transform of a causal signal $g(t)$ is given by

$$G(s) = \frac{s}{(s^2+9)(s+2)}, \qquad \mathrm{Re}(s) > 0.$$

Determine $g(t)$.

Solution

a) 2.5p

$$\begin{aligned} F(s) &= \frac{1}{s} - \frac{s}{(s+2)^2+9} + \frac{1}{(s+2)^2+9} \\ &= \frac{1}{s} - \frac{s+2}{(s+2)^2+9} + \frac{3}{(s+2)^2+9} \end{aligned}$$

Standard inverse Laplace transform gives:

$$f(t) = \{1 - e^{-2t}[\cos(3t) - \sin(3t)]\}u(t)\,.$$

b) 2.5p

$$G(s) = -\frac{2}{13}\frac{1}{s+2} + \frac{2}{13}\frac{s}{s^2+9} + \frac{3}{13}\frac{3}{s^2+9}$$

Standard inverse Laplace transform gives:

$$g(t) = \left[-\frac{2}{13}e^{-2t} + \frac{2}{13}\cos(3t) + \frac{3}{13}\sin(3t)\right]u(t)\,.$$

Question 2 (7 points)

A signal $x(t)$ satisfies the differential equation

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{b^2}{4m}x(t) = 0 \qquad \text{for } t > 0,$$

where $m > 0$ and $b > 0$ are constants and the initial conditions are given by $x(0) = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t}(0) = v_0$.

a) Use the one-sided Laplace transform to determine the signal $x(t)$ for $t > 0$.

A signal $y(t)$ satisfies the differential equation

$$t\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} + ty(t) = 0 \qquad \text{for } t > 0,$$

with $y(0) = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}t}(0) = 0$.

b) Assuming that $y(t)$ has a Laplace transform, show that it is given by

$$Y(s) = \frac{A}{\sqrt{s^2+1}},$$

where A is a constant.

c) Determine the constant A.

Solution

a) 3p Apply a one-sided Laplace transform and take the initial conditions into account to obtain

$$\left(s^2 + \frac{b}{m}s + \frac{b^2}{4m}\right) X(s) = v_0.$$

Since $s^2 + \frac{b}{m}s + \frac{b^2}{4m} = (s + \frac{b}{2m})^2$, we have

$$X(s) = \frac{v_0}{(s+\frac{b}{2m})^2}.$$

Inverse Laplace transform gives $x(t) = v_0 t e^{-\frac{b}{2m}t}u(t)$ (critically damped harmonic oscillator).

b) 3p Apply a one-sided Laplace transform and take the initial conditions into account to obtain

$$\frac{1}{Y(s)}\frac{\mathrm{d}Y(s)}{\mathrm{d}s} = -\frac{s}{s^2+1}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}s}\ln Y(s) = -\frac{s}{s^2+1} = \frac{\mathrm{d}}{\mathrm{d}s}\left[-\frac{1}{2}\ln(s^2+1)\right]$$

from which it follows that

$$\ln Y(s) = \ln(s^2+1)^{-1/2} + \ln A = \ln\frac{A}{\sqrt{s^2+1}},$$

where A is a constant. From the above equation we observe that

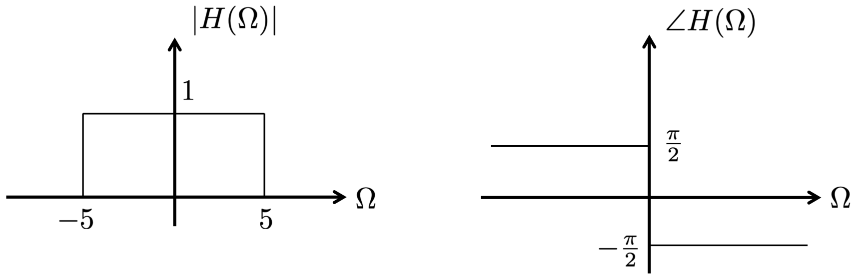
$$Y(s) = \frac{A}{\sqrt{s^2+1}}.$$

c) 1p Use the initial value theorem:

$$\lim_{s\to\infty} sY(s) = A\lim_{s\to\infty}\frac{s}{\sqrt{s^2+1}} = A = y(0) = 1.$$

$A = 1$.

Question 3 (6 points)



Consider a continous-time LTI system of which the frequency response is shown above.

a) Calculate the impulse response of the system.

Assume that the input of the filter is a periodic signal $x(t)$ having a Fourier series representation

$$x(t) = \sum_{k=1}^{\infty} \frac{2}{k^2} \cos(2kt).$$

b) What can you say about the continuity and differentiability of the signal $x(t)$?

c) Indicate how the series converges to $x(t)$ (point-wise, in norm, etc). Motivate your answer.

d) Determine the steady-state response of the system.

Solution

a) 2p

$$\begin{aligned} h(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\Omega) e^{j\Omega t} \mathrm{d}\Omega \\ &= \frac{1}{2\pi} \int_{-5}^{0} e^{j(\Omega t + \frac{\pi}{2})} \mathrm{d}\Omega + \frac{1}{2\pi} \int_{0}^{5} e^{j(\Omega t - \frac{\pi}{2})} \mathrm{d}\Omega \\ &= \frac{1}{j2\pi t} \left. e^{j(\Omega t + \frac{\pi}{2})} \right|_{-5}^{0} + \frac{1}{j2\pi t} \left. e^{j(\Omega t - \frac{\pi}{2})} \right|_{0}^{5} \\ &= \frac{1}{\pi t} \left(1 + \sin\left(5\Omega t - \frac{\pi}{2}\right)\right) \\ &= \frac{1}{\pi t} \left(1 - \cos(5\Omega t)\right). \end{aligned}$$

b) 1p The decay of the spectrum is $\mathcal{O}(1/k^2)$ from which we conclude that $x \in C^0$ (continuous but not differentiable).

c) 1p Since $x(t)$ is continuous and satisfies the Dirichlet conditions, the series converges point-wise (and thus in norm).

d) 2p

$$\begin{aligned} y_{\mathrm{ss}}(t) &= \sum_{k=1}^{\infty} H(2k) \frac{2}{k^2} \cos(2kt) \\ &= 2\cos\left(2t - \frac{\pi}{2}\right) + \frac{1}{2} \cos\left(4t - \frac{\pi}{2}\right) \\ &= 2\sin(2t) + \frac{1}{2} \sin(4t). \end{aligned}$$

Question 4 (5 points)

For a causal LTI system the response to an input signal $x = [\cdots, 0, \boxed{1}, 3, 3, 1, 0, \cdots]$ is given by $y = [\cdots, 0, \boxed{1}, 4, 6, 4, 1, 0, \cdots]$.

a) How are $x[n]$, $y[n]$ and the impulse response $h[n]$ of the system related? (Give a generic equation.)

b) Determine the impulse response $h[n]$ of the system. (Hint: first determine the filter length.)

c) Determine the z-transform $X(z)$ and $Y(z)$ of $x[n]$ and $y[n]$ (also specify the ROC).

d) Compute $H(z)$ and verify your answer under b).

Solution

a) 1p $y[n] = \sum_{k=0}^{L-1} h[k]x[n-k]$, where L is the filter length.

b) 2p The length of $x[n]$ is 4, and of $y[n]$ is 5. Thus, the filter length is $L = 2$. Subsequently, the convolution expressions are worked out as

$$h[0]x[n] + h[1]x[n-1] = y[n] \quad \Leftrightarrow \left\{ \begin{array}{rcll} h[0]\cdot 1 + h[1]\cdot 0 & = & 1 & \Rightarrow h[0] = 1 \\ h[0]\cdot 3 + h[1]\cdot 1 & = & 4 & \Rightarrow h[1] = 1 \\ h[0]\cdot 3 + h[1]\cdot 3 & = & 6 & \Rightarrow \text{(check)} \\ h[0]\cdot 1 + h[1]\cdot 3 & = & 4 & \Rightarrow \text{(check)} \\ h[0]\cdot 0 + h[1]\cdot 1 & = & 1 & \Rightarrow \text{(check)} \end{array} \right.$$

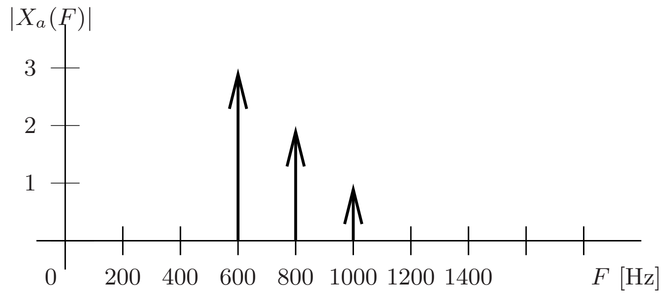
so that $h[n] = [\boxed{1}, 1, 0, \cdots]$.

c) 1p $X(z) = 1 + 3z^{-1} + 3z^{-2} + z^{-3}$, ROC: $\{z \neq 0\}$.
$Y(z) = 1 + 4z^{-1} + 6z^{-2} + 4z^{-3} + z^{-4}$, ROC: $\{z \neq 0\}$.

d) 1p In general: $Y(z) = H(z)X(z)$. Here we can easily see $Y(z) = X(z) + z^{-1}X(z)$, thus $H(z) = 1 + z^{-1}$, which is the z-transform of $h[n]$.

Question 5 (5 points)

The real-valued continuous-time signal $x_a(t)$ has frequency components as indicated below; the spectrum is real-valued.



a) What is the (minimal) sampling frequency required to avoid aliasing?

b) The signal is sampled at $F_s = 1000$ Hz, resulting in $x[n]$, there is no filtering. What frequency components are present in the sampled signal?

c) Draw the amplitude spectrum of $x[n]$; clearly indicate the frequencies and amplitudes.

Solution

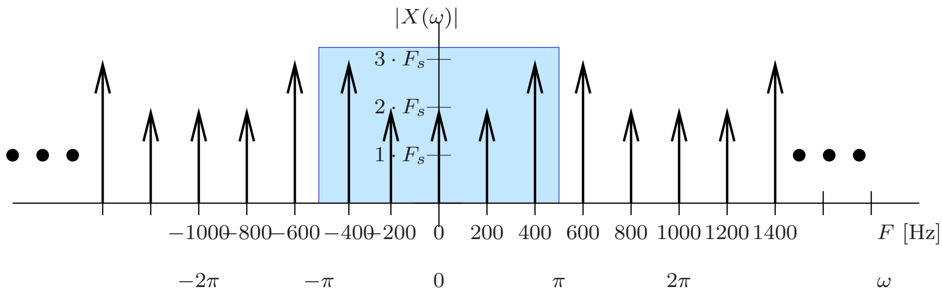
a) 1p Twice the highest frequency: $F_s = 2000$ Hz.

b) 2p The aliasing results in frequency components at all multiples of 1000 Hz. Also consider the negative frequencies! (The signal is real so the spectrum is symmetric.) We only need to consider components between -500 and 500 Hz, outside this fundamental interval the spectrum is repeated. This results in:

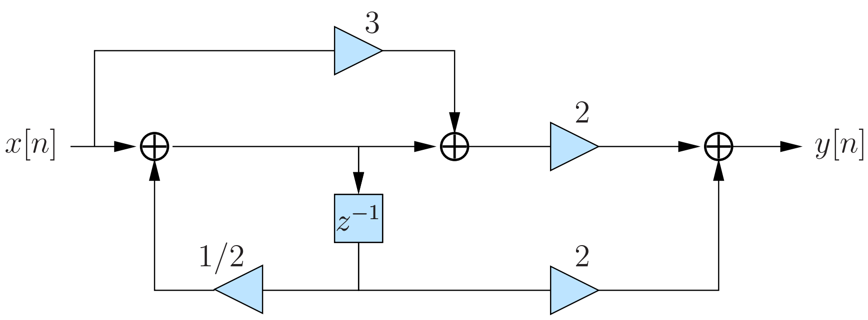
$$\begin{array}{ll} +600 \Rightarrow & -400 \\ -600 \Rightarrow & +400 \\ +800 \Rightarrow & -200 \\ -800 \Rightarrow & +200 \\ +1000 \Rightarrow & 0 \\ -1000 \Rightarrow & 0 \end{array}$$

The two components at 0 Hz add up, doubling the amplitude.

c) 2p Regarding the amplitudes, note that they get scaled by F_s. The fundamental interval runs from -500 Hz to 500 Hz, outside this interval it is periodic.



Question 6 (6 points)



a) Determine the transfer function $H(z)$ of the causal system shown above.

b) Determine its impulse response $h[n]$.

c) Is this a minimal system? (why)

d) Is this a stable system? (why)

e) Draw the "Direct form II" realization.

Solution

a) 2p Denote the signal at the input of the delay element equal to $P(z)$. Then

$$\left\{\begin{array}{lcl} P(z) &=& X(z)+\frac{1}{2}z^{-1}P(z) \\ Y(z) &=& 6X(z)+2P(z)+2z^{-1}P(z) \end{array}\right. \Rightarrow \left\{\begin{array}{lcl} P(z) &=& X(z)\dfrac{1}{1-\frac{1}{2}z^{-1}} \\ Y(z) &=& X(z)(6+\dfrac{2}{1-\frac{1}{2}z^{-1}}+\dfrac{2z^{-1}}{1-\frac{1}{2}z^{-1}}) \\ &=& =\dfrac{8-z^{-1}}{1-\frac{1}{2}z^{-1}} \end{array}\right.$$

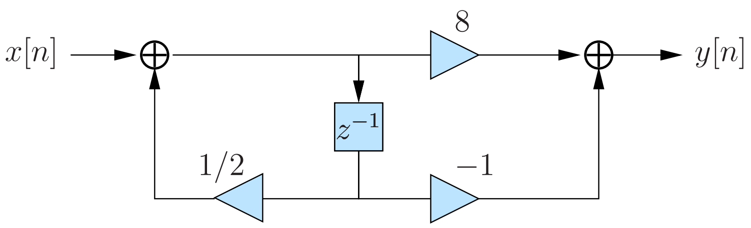
b) 1p The corresponding impulse response is

$$h[n] = 8(\frac{1}{2})^n u[n] - (\frac{1}{2})^{n-1}u[n-1]$$

c) 1p Yes (system order equal to the number of delays).

d) 1p Yes (causal system with pole $z = 1/2$ within the unit circle)

e) 1p



Question 7 (6 points)

We would like to design a *first-order* digital lowpass filter with the following specifications:

- Passband: until 7.2 kHz
- Damping outside the passband: at least 10 dB
- Sample rate: 48 kHz

The digital filter will be designed by applying the bilinear transform to an analog transfer function. We will use a Butterworth filter. The expression for the frequency response of a prototype n-th order low-pass Butterworth filter is

$$|H_a(\Omega)|^2 = \frac{1}{1+\epsilon^2(\Omega/\Omega_p)^{2n}} \, .$$

a) What is the passband frequency (in rad) in the digital time domain?
b) What is the passband frequency of the analog lowpass filter?
c) What is the frequency response $|H_a(\Omega)|^2$ for the analog filter that satisfies the specifications?
d) What is the transfer function $H_a(s)$ for that analog filter?
e) What is $H(z)$?
f) Demonstrate (verify) that the design of $H(z)$ satisfies the specifications.

Note: if you get stuck at some point, then make a reasonable assumption so you can continue with the rest of the questions.

Solution

a) 1p $\omega_p = \frac{7200}{48000} 2\pi = 0.3\pi$.

b) 1p The bilinear transform gives the mapping $\Omega_p = \tan(\omega_p/2) = 0.5095$.

c) 1p For $n = 1$ we have $|H_a(\omega)|^2 = \dfrac{1}{1 + \epsilon^2(\Omega/\Omega_p)^2}$.

Determine ϵ from the requirement on the damping at $\omega_p = 0.3\pi$. From $|H(\omega = 0.3\pi)|^2 = 0.1$ (damping 10 dB) we have

$$\frac{1}{1+\epsilon^2} = 0.1 \qquad \Rightarrow \qquad \epsilon = 3$$

$|H_a(\omega)|^2 = \dfrac{1}{1 + 9(\Omega/0.3\pi)^2}$.

d) 1p $|H(\omega)|^2 = H(s)H(-s)\big|_{s=j\Omega} = \dfrac{1}{1 - \epsilon^2(s/\Omega_p)^2}\big|_{s=j\Omega}$ results in $H(s) = \dfrac{1}{1 + \epsilon s/\Omega_p}$

Hence $H(s) = \dfrac{1}{1 + \epsilon/\Omega_p \cdot s} = \dfrac{1}{1 + 5.88 \cdot s}$

e) 1p Use the bilinear transform:

$$s \to \frac{1 - z^{-1}}{1 + z^{-1}}$$

resulting in

$$H(z) = \frac{1}{1 + 5.88\frac{1-z^{-1}}{1+z^{-1}}} = \frac{1}{6.88} \cdot \frac{1 + z^{-1}}{1 - 0.7096 \cdot z^{-1}}$$

f) 1p Verify:

$$|H(\omega = 0)| = |H(z^{-1} = 1)| = 1$$
$$|H(\omega = \pi)| = |H(z^{-1} = -1)| = 0$$
$$|H(\omega = 0.3\pi)|^2 = |H(z^{-1} = 0.59 - j0.81)|^2 = \tfrac{1}{(6.88)^2} \tfrac{(1+0.59)^2 + (0.81)^2}{(1-0.7096\cdot 0.59)^2 + (0.7096\cdot 0.81)^2} = \cdots = 0.1$$