# Exam EE2S11 Signals and Systems
# Resit on complete course: 24 July 2018, 13:30–16:30

Closed book; one double-sided A4 page of handwritten notes permitted

This exam consists of seven questions (40 points)

**Question 1 (4 points)**

Given the signal $f(t) = \begin{cases} 0, & \text{for } t < 0, \\ 2, & \text{for } t = 0, \\ 4, & \text{for } 0 < t < 2, \\ 3, & \text{for } t = 2, \\ 2, & \text{for } t > 2. \end{cases}$

a) Express $f(t)$ in terms of two unit step functions.

b) Determine the Laplace transform of $f(t)$ including its ROC.

c) Determine the derivative $g(t) = \frac{\mathrm{d}f}{\mathrm{d}t}$.

d) Determine the Laplace transform of $g(t)$ including its ROC.

**Answer:**

a) $f(t) = 4u(t) - 2u(t-2)$.

b) $F(s) = 2\dfrac{2 - e^{-s}}{s}, \quad \text{ROC} = \{s \in \mathbb{C}; \text{Re}(s) > 0\}$.

c) $g(t) = 4\delta(t) - 2\delta(t-2)$.

d) $G(s) = 2(2 - e^{-s}) = sF(s)$, ROC $= \mathbb{C}$.

**Question 2 (4 points)**

Given the signal $f(t) = \begin{cases} 0, & \text{for } t < 0, \\ \sin(t), & \text{for } 0 \le t \le \pi, \\ 0, & \text{for } t > \pi. \end{cases}$

a) Determine the Laplace transform of $f(t)$ including its ROC.

Now consider a signal $g(t)$ that vanishes for $t < 0$ and is periodic with a period $2\pi$ for $t > 0$, that is, $g(t + 2\pi) = g(t)$ for $t > 0$. On the interval $0 \le t \le 2\pi$, the signal $g(t)$ is given by

$$g(t) = \begin{cases} \sin(t) & \text{for } 0 \le t \le \pi, \\ 0 & \text{for } \pi < t \le 2\pi. \end{cases}$$

b) Determine the Laplace transform of $g(t)$ including its ROC.

**Answer:**

a) $F(s) = \dfrac{1+e^{-s\pi}}{s^2+1}, \quad \text{ROC} = \mathbb{C}.$

No poles at $s = \pm \text{j}$.

b) $G(s) = \dfrac{1}{1-e^{-s2\pi}} \dfrac{1+e^{-s\pi}}{s^2+1}, \quad \text{ROC} = \{s \in \mathbb{C}; \text{Re}(s) > 0\}.$

**Question 3 (7 points)**

Consider the $2\pi$-periodic signal $x(t)$, of which one period is described by $x(t) = \begin{cases} 1, & 0 \le t < \pi, \\ 0, & \pi \le t < 2\pi. \end{cases}$

a) Sketch the signal $x(t)$ (show at least three periods of $x$).

The Fourier series representation of a $2\pi$-periodic signal $x(t)$ is given by

$$x(t) = c_0 + 2\sum_{k=0}^{\infty} \left(c_k \cos(kt) + d_k \sin(kt)\right), \quad \text{where} \quad \begin{cases} c_k &= \dfrac{1}{2\pi}\displaystyle\int_{t_0}^{t_0+2\pi} x(t)\cos(kt)dt, \quad k = 0, 1, 2, \cdots \\ d_k &= \dfrac{1}{2\pi}\displaystyle\int_{t_0}^{t_0+2\pi} x(t)\sin(kt)dt, \quad k = 1, 2, \cdots \end{cases}$$

b) Show that the Fourier coefficients $c_k$ and $d_k$ of the signal $x(t)$ specified above are given by

$$c_k = \begin{cases} 0, & k \ne 0, \\ \frac{1}{2}, & k = 0, \end{cases} \quad \text{and} \quad d_k = \frac{1}{2\pi k}\left(1 - (-1)^k\right) \text{ for all } k.$$

We can also express the signal $x(t)$ into its complex Fourier series representation, given by

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jkt}, \quad \text{where} \quad X_k = \frac{1}{2\pi}\int_{t_0}^{t_0+2\pi} x(t)e^{-jkt}dt, \quad k \in \mathbb{Z}.$$

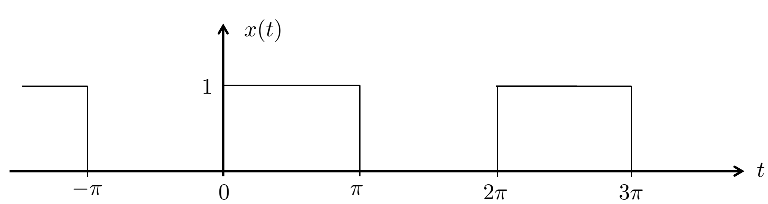
c) What is the relation between the coefficients $c_k, d_k$ and the coefficients $X_k$? Motivate your answer.

d) The Fourier coefficients $X_k$ in this example are purely imaginary, except for $k = 0$, for which $X_k$ is real. What is the reason that $X_0$ is real valued and that $X_k$ is imaginary for $k \ne 0$ ? Motivate your answer.

e) The magnitude and phase of the Fourier coefficients $X_k$ in this example are even and odd symmetric, respectively. That is, $|X_k| = |X_{-k}|$ and $\angle X_k = -\angle X_{-k}$. What is the reason for these specific symmetry properties? Motivate your answer.

**Answer:**

a)



b) We have

$$c_0 = \frac{1}{2\pi}\int_0^{\pi} dt = \frac{1}{2}.$$

Since $x(t) - c_0 = x(t) - \frac{1}{2}$ is odd-symmetric, the remaining $c_k$s are all zero. The coefficients $d_k, k = 1, 2, \ldots,$ are given by

$$d_k = \frac{1}{2\pi}\int_0^{\pi} \sin(kt)dt = \frac{-1}{2\pi k} \left.\cos(kt)\right|_0^{\pi} = \frac{1}{2\pi k}\left(1 - (-1)^k\right).$$

c) We have

$$\begin{aligned} X_k &= \frac{1}{2\pi}\int_{t_0}^{t_0+2\pi} x(t)e^{-jkt}dt \\ &= \frac{1}{2\pi}\int_{t_0}^{t_0+2\pi} x(t)\,)(\cos(kt) - j\sin(kt))\,dt = c_k - jd_k. \end{aligned}$$

Similarly, we find that $X_{-k} = c_k + jd_k$ so that $2c_k = X_k + X_{-k}$ and $2d_k = j(X_k - X_{-k})$.

d) The coefficients $X_k$ kan be computed as $X_k = c_k - jd_k$. Since all $c_k$s are zero, except for $c_0$ wich is $\frac{1}{2}$, we conclude that the Fourier coefficients $X_k$ in this example are purely imaginary, except for $k = 0$, for which $X_k$ is real.

This result can, of course, also be derived by directly computing the $X_k$s from the definition of the complex Fourier transform. Indeed, we have

$$X_k = \frac{1}{2\pi}\int_0^{\pi} e^{-jkt}dt = \begin{cases} \frac{-1}{j2\pi k} \left. e^{-jkt}\right|_0^{\pi} = \frac{1-(-1)^k}{j2\pi k}, & k \neq 0, \\ \frac{1}{2}, & k = 0. \end{cases}$$

e) This is because $x(t)$ is real-valued. Indeed, if $x(t)$ is real-valued we have

$$X_k^* = \frac{1}{2\pi}\int_{t_0}^{t_0+2\pi} x(t)e^{jkt}dt = X_{-k}.$$

That is, the spectrum is conjugate symmetric and we have $|X_k|e^{-j\angle X_k} = |X_{-k}|e^{j\angle X_{-k}}$ from which we conclude that $|X_{-k}| = |X_k|$ and $\angle X_{-k} = -\angle X_k$.

**Question 4 (6 points)**

a) Determine the $z$-transform for the following discrete-time signal, also specify the ROC:

$$x[n] = \frac{1}{2}\left(1 + (-1)^n\right)u[n].$$

b) Let $x[n] = \alpha^n u[n]$, where $\alpha < 1$. Determine the convolution $r[n] = x[n] * x[-n]$.

c) Assuming $x[n]$ is causal, determine the signal $x[n]$ corresponding to the $z$-transform:

$$X(z) = \frac{z^{-2} + 10z^{-1}}{z^{-2} + 6z^{-1} + 8}.$$

d) Determine the signal $x[n]$ corresponding to the DTFT

$$X(e^{j\omega}) = e^{j\pi/4}\delta(\omega - 1) + e^{-j\pi/4}\delta(\omega + 1).$$

**Answer:**

a) $X(z) = \dfrac{1}{1 - z^{-2}}, \quad |z| > 1.$

b) $R(z) = \dfrac{1}{1 - \alpha z^{-1}} \dfrac{1}{1 - \alpha z} = \dfrac{1}{1 - \alpha^2} \left[ \dfrac{1}{1 - \alpha z^{-1}} + \dfrac{\alpha z}{1 - \alpha z} \right]$, with ROC $\alpha < |z| < \frac{1}{\alpha}$.

$$r[n] = \frac{1}{1 - \alpha^2} \left[ \alpha^n u[n] + \alpha^{-n} u[-n-1] \right] = \frac{\alpha^{|n|}}{1 - \alpha^2}.$$

c)

$$X(z) = \frac{z^{-2} + 10z^{-1}}{z^{-2} + 6z^{-1} + 8} = 1 + \frac{4z^{-1} - 8}{z^{-2} + 6z^{-1} + 8} = 1 - \frac{8}{z^{-1} + 2} + \frac{12}{z^{-1} + 4} = 1 - \frac{4}{1 + \frac{1}{2}z^{-1}} + \frac{3}{1 + \frac{1}{4}z^{-1}}$$

$$x[n] = \delta[n] - 4\left(-\frac{1}{2}\right)^n u[n] + 3\left(-\frac{1}{4}\right)^n u[n].$$

Since $x[0] = 0$, we can simplify to

$$x[n] = -4\left(-\frac{1}{2}\right)^n u[n-1] + 3\left(-\frac{1}{4}\right)^n u[n-1].$$

Alternatively, in the first step write

$$X(z) = z^{-1} \cdot \frac{z^{-1} + 10}{z^{-2} + 6z^{-1} + 8} = z^{-1}\left(\frac{4}{z^{-1} + 2} - \frac{3}{z^{-1} + 4}\right) = z^{-1}\left(\frac{2}{1 + \frac{1}{2}z^{-1}} - \frac{3/4}{1 + \frac{1}{4}z^{-1}}\right)$$

resulting in

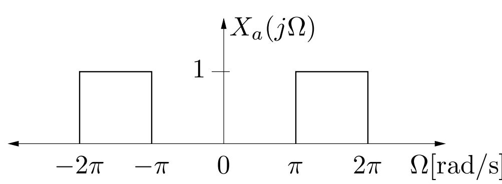
$$x[n] = 2\left(-\frac{1}{2}\right)^{n-1} u[n-1] - \frac{3}{4}\left(-\frac{1}{4}\right)^{n-1} u[n-1]$$

This is the same as the previous answer.

d) $\frac{1}{\pi} \cos(n + \frac{\pi}{4})$.
Obtained by using $e^{j\omega_0 n} \leftrightarrow 2\pi\delta(\omega - \omega_0)$, hence $e^{j\pi/4}\delta(\omega - 1) \leftrightarrow \frac{1}{2\pi} e^{j(n+\pi/4)}$.

## Question 5 (6 points)

The continuous-time signal $x_a(t)$ has a (real-valued) Fourier transform $X_a(j\Omega)$ as shown here:




a) Is this a bandlimited signal? What is its maximal frequency (in Hz)?
b) At what sampling period $T_s$ should we sample this signal, avoiding aliasing?

We sample the signal at $T_s = \frac{1}{2}$ sec, and obtain the discrete-time signal $x[n]$. The DTFT spectrum of $x[n]$ is $X(e^{j\omega})$.

c) How is $\omega$ related to $\Omega$?
d) Draw $X(e^{j\omega})$; also indicate the values on both axes.

Consider now the signal $y_a(t) = (x_a(t))^2$.

e) Make a plot of $Y_a(j\Omega)$; also indicate the values on both axes.

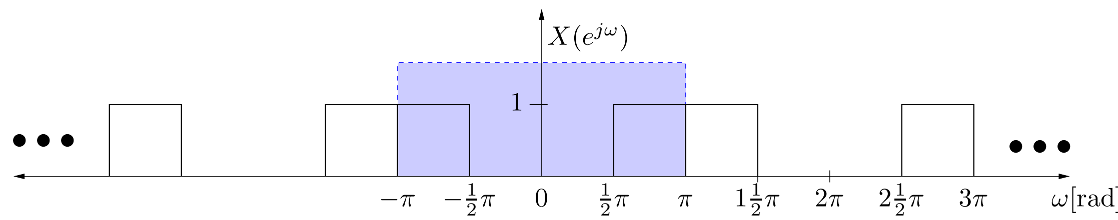
**Answer:**

a) Yes; since $\Omega = 2\pi F$, the maximal frequency is $F = 1$ Hz.

b) The sample rate should be at least $F_s = 2$ Hz, so $T_s \leq \frac{1}{2}$ s.

c) $\omega = \Omega T_s$.

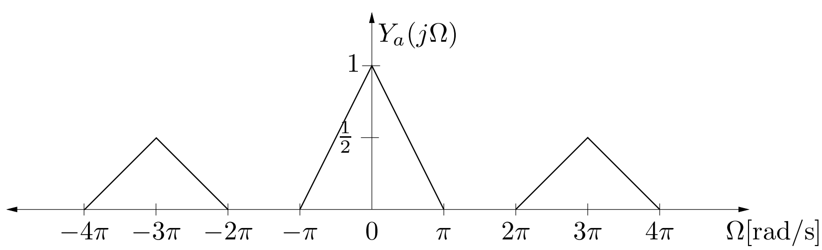
d)




e) $y_a(t) = x_a^2(t) = x_a(t) \cdot x_a(t) \quad \leftrightarrow \quad Y(j\Omega) = \frac{1}{2\pi} X_a(j\Omega) * X_a(j\Omega)$.

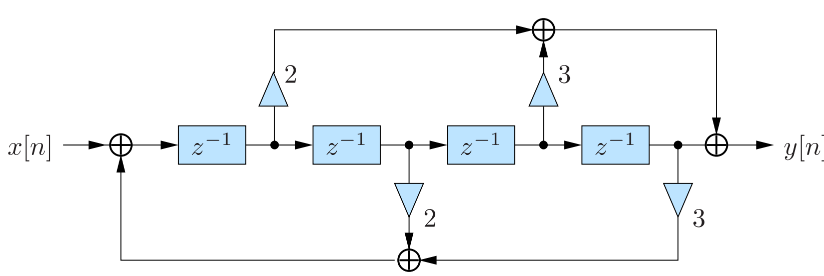
As $X_a(j\Omega)$ consists of two "bricks", the convolution produces 4 terms. Each has a triangular shape (cf. the convolution of a brick with itself). Two of the terms will center at $\Omega = 0$ hence the amplitude will be double at that place (as we assumed a real-valued spectrum here). At $\Omega = 0$, the convolution $X_a(j\Omega) * X_a(j\Omega)$ will be equal to the energy in the brick, which is $\pi$. Thus, the resulting amplitude of the triangle is $2\pi/2\pi = 1$.

Note that the width of each triangle is twice the width of the brick. Thus, we would have to sample the squared signal at twice the rate needed for $x_a(t)$.




### Question 6 (6 points)

Consider the following system realization:




a) Determine the transfer function $H(z)$.

b) Determine all poles and zeros of $H(z)$.

c) Is this a stable system? (Why?)

d) Is this a minimal realization? (Why?)

**Answer:**

a) $H(z) = \dfrac{2z^{-1} + 3z^{-3}}{1 - 2z^{-2} - 3z^{-4}} = \dfrac{2z^3 + 3z}{z^4 - 2z^2 - 3}$

b) Poles: $(z^2 - 3)(z^2 + 1) = 0$ so that $z_{1,2} = \pm\sqrt{3}$, $z_{3,4} = \pm j$.

Zeros: unintentionally, this realization was in a form that makes this too hard to compute without a computer. Nonetheless, you can note that there is a zero at $z = \infty$.

c) Not stable, poles outside the unit circle.

d) Minimal: 4th order and used 4 delays (in fact it is in Direct form no. II).

**Question 7 (7 points)**

We would like to design a digital *high*pass filter with the following specifications:

– Passband: starting at 3.0 kHz; ripple in the passband: $\leq 1$ dB
– Stopband: below 2.0 kHz; stopband damping: $\geq 40$ dB
– Sample rate: 12 kHz

The digital filter will be designed by applying the bilinear transform on an analog transfer function. We will use a Chebyshev filter. The expression for the frequency response of a prototype $n$-th order low-pass Chebyshev filter is

$$|H(\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_n^2(\Omega)} .$$

a) What are the passband and stopband frequencies (in rad) in the digital time domain?

b) What are the filter specifications in the analog time domain?

c) Which transformation will you use to transform a lowpass into a highpass filter? What are the specifications of the analog lowpass filter?

d) Compute the required filter order $n$ for a Chebyshev filter.
(Remark: $\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$.)

e) Suppose that the resulting Chebyshev lowpass filter has this form:

$$H(s) = \frac{B(s)}{A(s)} = \frac{1}{a_0 + a_1 s + \cdots + a_n s^n} .$$

How can you find the filter coefficients of the digital *high*pass filter?

**Answer:**

a)

$$\omega_p = \frac{3}{12} 2\pi = \frac{1}{2}\pi , \qquad \omega_s = \frac{2}{12} 2\pi = \frac{1}{3}\pi$$

b)

$$\text{passband: } \Omega_p = \tan(\frac{\omega_p}{2}) = 1 , \qquad \text{stopband: } \Omega_s = \tan(\frac{\omega_p}{2}) = .5774$$

with the same damping specs as before.

c)

$$s \to \frac{1}{s}, \qquad \Omega \to \frac{1}{\Omega}$$

(There are other options, more generally $s \to \frac{\Omega_0}{s}$, you will have to take this into account later in item e.)

Specs for the lowpass filter:

- $\Omega'_p = 1$, ripple in passband smaller than 1 dB
- $\Omega'_s = 1/0.5774 = 1.7321$, damping in stopband larger than 40 dB

d) Passband (using $T_n^2(1) = 1$ for any $n$):

$$\frac{1}{1+\epsilon^2} = \delta_p^2 = (10^{-1/20})^2 \quad \Rightarrow \quad \epsilon = 0.5087$$

Stopband:

$$\frac{1}{1+\epsilon^2 T_n^2(\Omega'_s/\Omega'_p)} = \delta_s^2 = (10^{-40/20})^2 = 10^{-4} \quad \Rightarrow \quad T_n^2(\Omega'_s/\Omega'_p) = \frac{10^4-1}{\epsilon^2} = 38640$$

Next use (for $\Omega > 1$) that $T_n(\Omega) = \cosh(n \cosh^{-1}(\Omega))$. This results in

$$n = \frac{\cosh^{-1}(\sqrt{38640})}{\cosh^{-1}(1.7321)} = 5.2126$$

So we should take $n = 6$.

e) First apply the lowpass to highpass transform: $s \to \frac{1}{s}$, this results in

$$H_{HP}(s) = \frac{1}{A(\frac{1}{s})} = \frac{s^n}{a_0 s^n + \cdots + a_n}$$

Then insert the bilinear transform $s = \frac{1-z^{-1}}{1+z^{-1}}$, this results in

$$H_{HP}(z) = \frac{(1-z^{-1})^n}{a_0(1-z^{-1})^n + a_1(1-z^{-1})^{n-1}(1+z^{-1}) + \cdots + a_n(1+z^{-1})^n}$$

The filter coefficients are found by writing out the numerator and the denominator as a polynomial in $z^{-1}$.