Exam EE2S11 Signals and Systems
Resit on complete course: 25 juli 2016, 13:30–16:30

Closed book; one double-sided A4 page of handwritten notes permitted

This exam consists of seven questions (45 points)

Question 1 (4 points)

Which of the following statements are true? (Just provide the answers, no motivation necessary.)

a) If $x(t)$ is a causal signal, the Region of Convergence (ROC) of its Laplace transform lies to the right of a line through the rightmost singularity.

b) If $x(t)$ is an anti-causal signal, the ROC of its Laplace transform lies to the left of a line through the leftmost singularity.

c) If $x(t)$ is a noncausal signal and its Laplace transform exists, then its ROC is a strip in the s-plane parallel to the imaginary axis.

d) The ROC does not contain any singularities.

Solution:

All statements are true.

Question 2 (8 points)

Determine the inverse Laplace transform of the following functions:

a) $X(s) = \dfrac{2s+1}{s^2+9}$, ROC: $\mathrm{Re}(s) > 0$

b) $Y(s) = \dfrac{s^2+2}{s^3-s}$, ROC: $\mathrm{Re}(s) > 1$

c) $Z(s) = \dfrac{2}{s^2(s-1)}$, ROC: $\mathrm{Re}(s) > 1$

Solution:

a)
$$X(s) = 2\frac{s}{s^2+9} + \frac{1}{3}\frac{3}{s^2+9} \quad \mathrm{Re}(s) > 0.$$

An inverse Laplace transform gives $x(t) = [2\cos(3t) + \frac{1}{3}\sin(3t)]u(t)$.

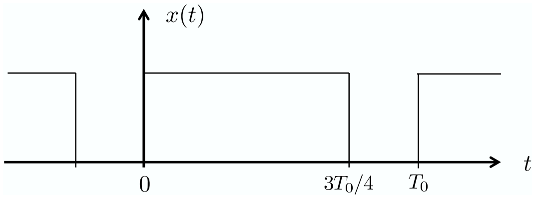
b) Since $s^3 - s = s(s+1)(s-1)$, we have
$$Y(s) = \frac{s^2+2}{s(s+1)(s-1)} = -2\frac{1}{s} + \frac{3}{2}\frac{1}{s-1} + \frac{3}{2}\frac{1}{s+1} \quad \mathrm{Re}(s) > 1.$$

An inverse Laplace transform gives $y(t) = (-2 + \frac{3}{2}e^{t} + \frac{3}{2}e^{-t})u(t)$.

c) We have
$$Z(s) = -2\frac{1}{s^2} - 2\frac{1}{s} + 2\frac{1}{s-1} \quad \mathrm{Re}(s) > 1.$$

An inverse Laplace transform gives $z(t) = 2(e^t - t - 1)u(t)$.

Question 3 (8 points)



Consider the T_0-periodic signal $x(t)$ as depicted above. The Fourier series of $x(t)$ can be expressed as

$$x(t) = c_0 + 2\sum_{k=1}^{\infty} \left(c_k \cos(k\Omega_0 t) + d_k \sin(k\Omega_0 t)\right), \qquad \Omega_0 = \frac{2\pi}{T_0},$$

where the coefficients c_k and d_k are given by

$$c_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos(k\Omega_0 t)dt, \quad d_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\sin(k\Omega_0 t)dt, \quad k = 0, 1, 2, \cdots$$

a) Compute the Fourier coefficients c_k and d_k.

b) The decay of the Fourier coefficients is $\mathcal{O}(1/k)$. Explain why this is the case.

c) What would be the effect on the Fourier coefficients c_k and d_k if we raise the signal $x(t)$ by a constant $c \in \mathbb{R}$? That is, if we consider the signal $x(t) + c$.

d) What would be the effect on the Fourier coefficients c_k and d_k if we shift the signal $x(t)$ by a constant $c \in \mathbb{R}$? That is, if we consider the signal $x(t - c)$.

e) Considering the shifted signal of item d), for what values of c do the coefficients c_k and/or d_k vanish? Motivate your answer.

Solution:

a) c_0 is given by

$$c_0 = \frac{1}{T_0}\int_0^{3T_0/4} dt = 3/4.$$

For $k = 1, 2, \ldots$ we have

$$\begin{aligned} c_k &= \frac{1}{T_0}\int_0^{3T_0/4} \cos(k\Omega_0 t)dt = \frac{1}{2\pi k}\sin(k\Omega_0 t)\Big|_0^{3T_0/4} \\ &= \frac{1}{2\pi k}\sin(k3\pi/2), \end{aligned}$$

and

$$\begin{aligned} d_k &= \frac{1}{T_0}\int_0^{3T_0/4} \sin(k\Omega_0 t)dt = \frac{-1}{2\pi k}\cos(k\Omega_0 t)\Big|_0^{3T_0/4} \\ &= \frac{1}{2\pi k}\left(1 - \cos(k3\pi/2)\right). \end{aligned}$$

b) The decay of the Fourier coefficients is $\mathcal{O}(1/k)$ because the signal is not continuous.

c) An offset c will only affect c_0, which represents the mean value. Hence $c_0' = c_0 + c$ and all the other c_k's and d_k's remain the same.

d) The value c_0 will not change. For the other c_k's and d_k's we find

$$c_k' = \frac{1}{T_0}\int_c^{3T_0/4+c} \cos(k\Omega_0 t)dt = \frac{1}{2\pi k}\left(\sin(k\Omega_0(3T_0/4+c)) - \sin(k\Omega_0 c)\right),$$

$$d_k' = \frac{1}{T_0}\int_c^{3T_0/4+c} \sin(k\Omega_0 t)dt = \frac{1}{2\pi k}\left(\cos(k\Omega_0 c) - \cos(k\Omega_0(3T_0/4+c))\right).$$

e) The coefficients d_k will vanish when the shifted function is even symmetric. This happens for $c = T_0/8 \bmod T$ and $c = -3T_0/8 \bmod T$. Since we can't make the shifted function oddly symmetric, we can't make all c_k's zeros.

Question 4 (10 points)

Determine the z-transform for the following discrete-time signals, also specify the ROC:

a) $x[n] = u[n] + \left(\frac{1}{2}\right)^n u[n-2]$,

b) $x[n] = 2\sin(\frac{\pi}{6}n)u[-n]$.

Determine the signals $x[n]$ corresponding to the z-transforms:

c) $X(z) = (1-z^{-1})^2, \quad$ ROC: $z \neq 0$.

d) $X(z) = \dfrac{1.2}{(1-z^{-1})(1+0.2z^{-1})}, \quad$ ROC: $0.2 < |z| < 1$.

Determine the frequency response $H(e^{j\omega})$ for the system defined by the difference equation:

e) $y[n] = 0.5y[n-1] + x[n] + x[n-1], \quad n \geq 0$.

Solution:

a) First rewrite the second term as $(\frac{1}{2})^2(\frac{1}{2})^{n-2}u[n-2]$, so that we can use the shift property of the z-transform. It follows that

$$X(z) = \frac{1}{1-z^{-1}} + \frac{1}{4}\frac{z^{-2}}{1-(\frac{1}{2})z^{-1}}, \qquad \text{ROC: } |z| > 1.$$

b) Use the property $\sin(\omega_0 n)u(n) \quad \Leftrightarrow \quad \frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$ (ROC: $|z| > 1$).

We also need to use $x[-n] \quad \Leftrightarrow \quad X(z^{-1})$ and note that $\sin(\omega_0 n) = -\sin(-\omega_0 n)$.

It follows that

$$X(z) = \frac{-z}{1-\sqrt{3}z+z^2} \qquad \text{ROC: } |z| < 1.$$

c) $X(z) = (1-z^{-1})^2 = 1 - 2z^{-1} + z^{-2}$. Hence

$$x[n] = \delta[n] - 2\delta[n-1] + \delta[n-2].$$

d) Partial fraction expansion gives

$$X(z) = \frac{1}{1-z^{-1}} + \frac{0.2}{1+0.2z^{-1}}.$$

Inspecting the ROC, we see that the first term will give an anti-causal signal and the second term will become causal. We therefore write

$$X(z) = \frac{-z}{1-z} + \frac{0.2}{1+0.2z^{-1}} .$$

Applying the inverse transform gives

$$x[n] = -u[-n-1] + 0.2(-0.2)^n u[n] .$$

e) We first determine $H(z)$ as

$$H(z) = \frac{1+z^{-1}}{1-\frac{1}{2}z^{-1}} .$$

Therefore, the frequency response is

$$H(e^{j\omega}) = \frac{1+e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}} .$$

Question 5 (4 points)

A continuous-time signal $x_a(t)$ has a Fourier transform $X_a(\Omega) = u(\Omega+1) - u(\Omega-1)$.

a) Determine $x_a(t)$.

b) What is the largest value of the sampling period T_s that would not cause aliasing when sampling $x_a(t)$?

c) We sample the signal at $T_s = \pi$. Draw the sampled signal $x[n]$ (also specify the values on the axes).

Solution:

a) Table on p.383 (with $\Omega_0 = 1$):

$$x_a(t) = \frac{\sin(t)}{\pi t}$$

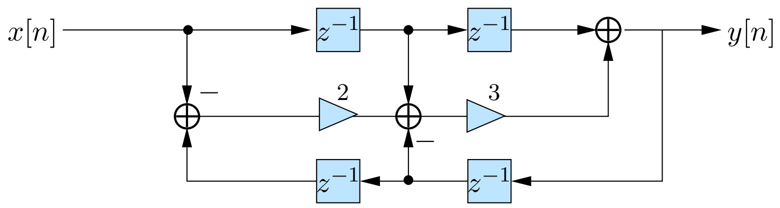
b) The signal is bandlimited with bandwidth $\Omega_{\max} = 1$, or $F_{\max} = 1/(2\pi)$. Therefore, we have to sample at Nyquist rate $F_s = 2F_{\max} = 1/\pi$. The corresponding sampling period is $T_s = \pi$.

c) In this case we sample the sin-function exactly at its zeros. Therefore

$$x[n] = \frac{1}{\pi}\delta[n]$$

Question 6 (4 points)

Consider the following system realization:



a) Determine the transfer function $H(z)$.

b) Is this a stable system? (Why?)

c) Is this a minimal realization? (Why?)

Solution:

a) Introduce additional variables $P(z) = z^{-1}X(z)$ at the output of the first delay, and $Q(z) = z^{-1}Y(z)$ at the output of the delay in the bottom branch. The equation for $Y(z)$ is

$$Y = z^{-1}P + 3(P + 2(z^{-1}Q - X) - Q)$$

Substitute $P(z)$ and $Q(z)$ to obtain

$$Y = z^{-2}X + 3(z^{-1}X + 2(z^{-2}Y - X) - z^{-1}Y)$$

$$(1 + 3z^{-1} - 6z^{-2})Y = (-6 + 3z^{-1} + z^{-2})X$$

The transfer function is

$$H(z) = \frac{-6 + 3z^{-1} + z^{-2}}{1 + 3z^{-1} - 6z^{-2}}$$

(This shows that it is a 2nd order allpass function.)

b) The poles are $z = -1\frac{1}{2} \pm j\frac{1}{2}\sqrt{15}$. Clearly the poles are outside the unit circle: unstable.

c) Not minimal because 4 delays are used and $H(z)$ is 2nd order.

Question 7 (7 points)

A second-order analog lowpass filter (Butterworth filter) has transfer function

$$G_a(s) = \frac{1}{s^2 + \sqrt{2}s + 1}\,.$$

The 3-dB cut-off frequency of this filter is $\Omega_c = 1$ rad/s.

Using the bilinear transform and the above filter as a template, we will now design a digital *high*-pass filter $H(z)$ with cut-off frequency $\omega_c' = \frac{2}{3}\pi$.

a) What should be the corresponding cut-off frequency in the analog frequency domain?

b) Which frequency transformation should be used?

c) What is $H_a(s)$?

d) What is $H(z)$?

e) Verify that the design meets the specifications.

Solution:

a) The bilinear transform gives $\Omega_c' = \tan(\frac{1}{2}\omega_c') = \tan(\frac{1}{3}\pi) = \sqrt{3}$.

b) $\Omega \to \dfrac{\sqrt{3}}{\Omega}$. This transforms lowpass into highpass, with cutoff at $\sqrt{3}$.

c) Likewise $s \to \dfrac{\sqrt{3}}{s}$. This gives

$$H_a(s) = \frac{1}{\frac{3}{s^2} + \sqrt{6}s + 1} = \frac{s^2}{s^2 + \sqrt{6}s + 3}$$

d) Apply the bilinear transform: $s \to \dfrac{1-z^{-1}}{1+z^{-1}}$. This gives

$$H(z) = \frac{(1-z^{-1})^2}{(1-z^{-1})^2 + \sqrt{6}(1-z^{-1})(1+z^{-1}) + 3(1+z^{-1})^2} = \cdots$$

e) We evaluate at $\omega = 0$, $\omega = \pi$ and $\omega = \omega_c$:

$$H(z = e^{j0}) = 0$$
$$H(z = e^{j\pi}) = \tfrac{2^2}{2^2+0+0} = 1$$
$$H(z = e^{j2\pi/3}) = H(z = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}) = H_a(s = j\sqrt{3}) = \tfrac{-3}{-3+j\sqrt{6\cdot 3}+3} = j\tfrac{1}{\sqrt{2}}$$

This shows that indeed $H(z)$ is a highpass filter with -3 dB point at ω_c.