Delft University of Technology
Faculty of Electrical Engineering, Mathematics and Computer Science

EE2S11 SIGNALS AND SYSTEMS

Part 1, 14 December 2017, 13:30 - 15:30

Closed book; two A4-size pages with handwritten notes permitted. Graphic calculators not permitted.

This exam has four questions (35 points)

Question 1 (8 points)

Let

$$p(t) = \begin{cases} 1 & \text{for } 0 < t < 1, \\ 0 & \text{elsewhere} \end{cases}$$

and consider the signal $x(t) = p(t) * p(t) * p(t)$, where the asterisk denotes convolution.

a) Determine $x(t)$ for $t > 3$.

The signal

$$f(t) = \begin{cases} 0 & \text{for } t < 0, \\ \sin(t) & \text{for } 0 \leq t < \pi, \\ \pi/2 & \text{for } t = \pi, \\ t & \text{for } t > \pi, \end{cases}$$

can be written as

$$f(t) = \sin(t)u(t) + g(t-\pi)u(t-\pi),$$

where $u(t)$ is the unit step function.

b) Determine the signal $g(t)$.

c) Determine the one-sided Laplace transform of $f(t)$.

Solution

a) $x(t) = 0$ for $t > 3$ (support of x is the interval $(0, 3)$).

b)

$$\begin{aligned} f(t) &= \sin(t)u(t) + [t - \sin(t)]u(t-\pi) \\ &= \sin(t)u(t) + [\pi + (t-\pi) + \sin(t-\pi)]u(t-\pi). \end{aligned}$$

We conclude that $g(t) = \pi + t + \sin(t)$.

c) Using the above result, we immediately find

$$F(s) = \frac{1}{s^2+1} + e^{-\pi s}\left(\frac{\pi}{s} + \frac{1}{s^2} + \frac{1}{s^2+1}\right), \qquad \text{Re}(s) > 0.$$

Question 2 (9 points)

a) Find the one-sided Laplace transform (including its ROC) of the signal

$$x(t) = \sin(t)\cos(t)u(t),$$

where $u(t)$ is the Heaviside unit step function.

b) Find the one-sided Laplace transform (including its ROC) of the signal

$$y(t) = t^2 \sin(2t)u(t).$$

c) The one-sided Laplace transform of a causal signal $z(t)$ is given by

$$Z(s) = \frac{3s+9}{s^2+2s+10}, \qquad \text{Re}(s) > 0.$$

Determine $z(t)$.

Solution

a) $x(t) = \sin(t)\cos(t)u(t) = \frac{1}{2}\sin(2t)u(t)$. Its one-sided Laplace transform now easily follows as

$$X(s) = \frac{1}{2}\frac{2}{s^2+4} = \frac{1}{s^2+4}, \quad \text{Re}(s) > 0.$$

b)

$$Y(s) = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\frac{2}{s^2+4} = \frac{12s^2-16}{(s^2+4)^3}, \quad \text{Re}(s) > 0.$$

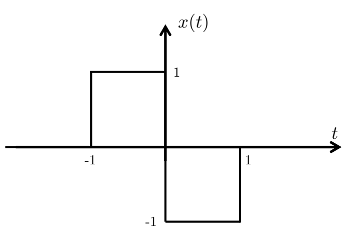
c)

$$Z(s) = \frac{3(s+3)}{(s+1)^2+9} = 3\frac{s+1}{(s+1)^2+9} + 2\frac{3}{(s+1)^2+9}, \quad \text{Re}(s) > 0.$$

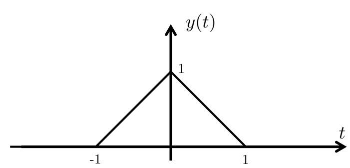
An inverse Laplace transform now gives $z(t) = e^{-t}[3\cos(3t) + 2\sin(3t)]u(t)$.

Question 3 (12 points)

Consider the continuous-time signals $x(t)$ and $y(t)$ depicted below:



a)



b)

The Fourier transform of $x(t)$ is given by

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \tag{1}$$

and similarly for $y(t)$.

a) Without explicitly computing the Fourier transform, what can you say about the spectrum of $x(t)$ and $y(t)$. Is it discrete or continuous, is it real, imaginary of complex-valued, and do we have symmetry in the spectrum? Motivate your answer.

b) Compute the Fourier transform of $x(t)$ using (1).

c) Compute the Fourier transform of $x(t)$ using the Laplace transform.

d) What is the value $X(0)$ and what information does this value give you about $x(t)$?

Now assume that the shown signal $x(t)$ is the input to an LTI system of which the output is given by the signal $y(t)$.

e) Compute the Fourier transform $Y(\Omega)$ of $y(t)$.

f) What is the order of decay of both spectra? Explain which one decays fastest and why.

g) Give a sketch of both $|X(\Omega)|$ and $|Y(\Omega)|$.

h) Show that the frequency response of the system is given by

$$H(j\Omega) = \frac{1}{j\Omega}, \quad \Omega \in \mathbb{R} \setminus \{0\}.$$

Solution

a) Since x and y are non-periodic signals, the spectrum is continuous. Since x is odd symmetric, the spectrum is purely imaginary, whereas the spectrum of y is real since y is even symmetric. Since both x and y are real, the spectra are conjugate symmetric. That is, the magnitude spectrum is even symmetric and the phase spectrum is odd symmetric.

b)

$$\begin{aligned} X(\Omega) &= \int_{-1}^{0} e^{-j\Omega t} dt - \int_{0}^{1} e^{-j\Omega t} dt \\ &= \frac{-1}{j\Omega} e^{-j\Omega t}\Big|_{-1}^{0} + \frac{1}{j\Omega} e^{-j\Omega t}\Big|_{0}^{1} \\ &= \frac{1}{j\Omega}\left(e^{j\Omega} + e^{-j\Omega} - 2\right). \\ &= \frac{2}{j\Omega}\left(\cos(\Omega) - 1\right). \end{aligned}$$

Note that X is indeed purely imaginary.

c) We have that $x(t) = u(t+1) - 2u(t) + u(t-1)$. Applying the Laplace transform yields

$$X(s) = \frac{1}{s}\left(e^{s} + e^{-s} - 2\right).$$

If the imaginary axis is contained in the ROC (which it is), the Fourier transform of x is given by

$$X(\Omega) = X(s)\Big|_{s=j\Omega} = \frac{1}{j\Omega}\left(e^{j\Omega} + e^{-j\Omega} - 2\right).$$

d) We have

$$X(\Omega) = \frac{e^{-j\Omega}}{j\Omega}\left(e^{2j\Omega} - 2e^{j\Omega} + 1\right) = \frac{e^{-j\Omega}}{j\Omega}\left(e^{j\Omega} - 1\right)^2.$$

Hence, $X(0) = 0$. This value is the average value (DC component) of the signal, which is indeed zero.

e) Since y is obtained by integrating x, its Laplace transform is given by

$$Y(s) = \frac{1}{s}X(s) = \frac{1}{s^2}\left(e^{s} + e^{-s} - 2\right),$$

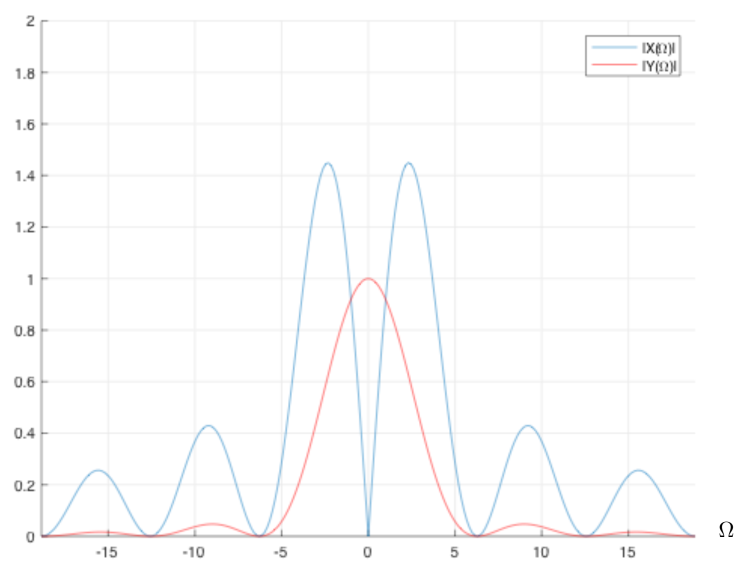
so that (the imaginary axis is in the ROC)

$$Y(\Omega) = \frac{-1}{\Omega^2}\left(e^{j\Omega} + e^{-j\Omega} - 2\right) = \frac{2}{\Omega^2}\left(1 - \cos(\Omega)\right).$$

Alternatively, we have $y(t) = r(t+1) - 2r(t) + r(t-1)$ and Laplace transformation of the indiviudual terms gives the same result. Note that Y is indeed real valued.

f) The decay of X is $\mathcal{O}(1/\Omega)$ since x has discontinuities. The decay of Y is faster since y doesn't have discontinuities. It is $\mathcal{O}(1/\Omega^2)$ because y is not differentiable.

g)



h) The frequency response of the filter is given by

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{j\Omega}, \quad \Omega \in \mathbb{R} \setminus \{0\}.$$

Question 4 (6 points)

Consider a causal LTI system with transfer function

$$H(s) = \frac{s}{(s+1)^2 + 3}, \quad s \in \text{ROC}.$$

a) Determine the poles and zeros of the system and draw them in the complex s-plane. Is the system BIBO stable? Motivate your answer.

b) Give an expression for the frequency, magnitude and phase response of the system.

c) Sketch the magnitude and phase response and indicate the values of $|H(j\Omega)|$ and $\angle H(j\Omega)$ for the (angular) frequencies $\Omega = 0, \Omega = \pm\sqrt{3}$ and $\Omega = \pm\infty$.

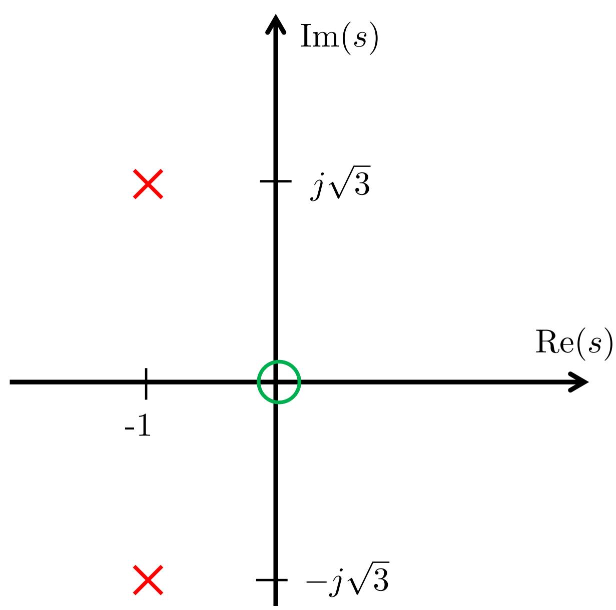
d) How should we modify the transfer function if we want to have a faster decay at $\Omega = \pm\infty$? Motivate your answer.

Solution

a) We have

$$H(s) = \frac{s}{(s+1-j\sqrt{3})(s+1+j\sqrt{3})}.$$

Hence, we have a zero at $s = 0$ and two poles at $s = -1 \pm j\sqrt{3}$. The causal system is BIBO stable since all poles lie in the left-hand plane.



b) The frequency response is given by

$$H(\Omega) = H(s)\Big|_{s=j\Omega} = \frac{j\Omega}{4-\Omega^2+2j\Omega}.$$

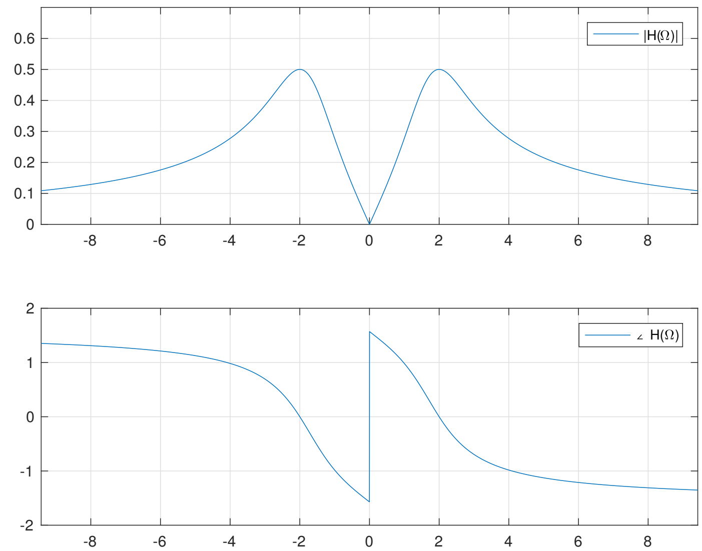
Hence, the magnitude response is given by

$$|H(\Omega)| = \frac{|\Omega|}{\sqrt{(4-\Omega^2)^2+4\Omega^2}}$$

and the phase response by

$$\angle H(\Omega) = \frac{\pi}{2}\mathrm{sign}(\Omega) - \tan^{-1}\left(\frac{2\Omega}{4-\Omega^2}\right).$$

c) We have $|H(0)| = 0, |H(\pm\sqrt{3})| = \sqrt{\frac{3}{13}}, |H(\pm\infty)| = 0$ and $\angle H(0^{\pm}) = \pm\frac{\pi}{2}, \angle H(\pm\sqrt{3}) = \pm(\frac{\pi}{2} - \tan^{-1}(2\sqrt{3})), \angle H(\pm\infty) = \mp\frac{\pi}{2}$.



d) In order to have faster decay, we can introduce additional poles.