Delft University of Technology
Faculty of Electrical Engineering, Mathematics and Computer Science

EE2S11 SIGNALEN EN SYSTEMEN

Part 1, 13 December 2016, 13:30 - 15:30

Closed book; two A4-size pages with handwritten notes permitted

This exam has four questions (40 points)

Question 1 (8 points)

Consider the standard triangular signal

$$\Lambda(t) = \begin{cases} 0 & t < 0, \\ t & 0 < t < 1, \\ 2-t & 1 < t < 2, \\ 0 & t > 2. \end{cases}$$

a) Determine the signal

$$f(t) = \frac{\mathrm{d}^2\Lambda(t)}{\mathrm{d}t^2}.$$

The ramp signal is given by $r(t) = t\,u(t)$ where $u(t)$ is the unit step signal

b) Write the triangular signal in terms of ramp signals.

c) Determine the two-sided Laplace transform $L(s)$ of $\Lambda(t)$.

d) Does $L(s)$ have a singularity at $s = 0$? (Motivate your answer.)

e) Determine, now using the Laplace transform, again the signal

$$f(t) = \frac{\mathrm{d}^2\Lambda(t)}{\mathrm{d}t^2}.$$

Solution

a) $f(t) = \delta(t) - 2\delta(t-1) + \delta(t-2)$.

b) $\Lambda(t) = r(t) - 2r(t-1) + r(t-2)$.

c) The Laplace transform of the ramp signal is given by $R(s) = 1/s^2$ with $\mathrm{Re}(s) > 0$. Using the translation property, we obtain

$$\begin{aligned} L(s) &= R(s) - 2e^{-s}R(s) + e^{-2s}R(s) \\ &= R(s)\left(1 - 2e^{-s} + e^{-2s}\right) \\ &= \frac{1}{s^2}\left(1 - 2e^{-s} + e^{-2s}\right), \qquad \mathrm{Re}(s) > 0. \end{aligned}$$

d) No, $L(s = 0) = 1$.

e) $F(s) = s^2 L(s) = 1 - 2e^{-s} + e^{-2s}$, $\mathrm{Re}(s) > 0$. The inverse Laplace transform gives $f(t) = \delta(t) - 2\delta(t-1) + \delta(t-2)$.

Question 2 (10 points)

The transfer function of an LTI system with one input and one output is given by

$$H(s) = \frac{s^2}{(s+10)^2}, \qquad \mathrm{Re}(s) > -10.$$

a) Determine the output signal $y(t)$ in case the input signal is given by $x(t) = u(t)$.

b) Determine the impulse response $h(t)$ of the system.

Now suppose that the LTI system consists of two causal parts, connected in series (cascade), without loading each other. The transfer functions of the parts are denoted by $H_1(s)$ and $H_2(s)$. The transfer function of part 1 is given by

$$H_1(s) = \frac{s}{s+10}, \qquad \mathrm{Re}(s) > -10.$$

c) Determine the transfer function $H_2(s)$ of part 2.

d) Determine the reponse *of part 1* on the non-causal input signal

$$x(t) = 3e^{-5|t|}.$$

Solution

a) $X(s) = 1/s$ for $\mathrm{Re}(s) > 0$.

$$Y(s) = H(s)X(s) = \frac{s}{(s+10)^2} = \frac{1}{s+10} - \frac{10}{(s+10)^2}, \qquad \mathrm{Re}(s) > 0.$$

The inverse Laplace transform gives $y(t) = (1 - 10t)e^{-10t}u(t)$.

b) Two solution approaches are given:

Method 1: The output signal $y(t)$ of part a) is the step response of the system. The impulse response is hence given by

$$h(t) = \frac{\mathrm{d}y}{\mathrm{d}t} = \delta(t) - 20(1 - 5t)e^{-10t}u(t).$$

Method 2: $x(t) = \delta(t)$, $X(s) = 1$ for $s \in \mathbb{C}$.

$$Y(s) = H(s)X(s) = \frac{s^2}{(s+10)^2} = 1 - 20\left[\frac{1}{s+10} - \frac{5}{(s+10)^2}\right],$$

for $\mathrm{Re}(s) > -10$. The inverse Laplace transform gives $y(t) = \delta(t) - 20(1 - 5t)e^{-10t}u(t)$.

c) The two subsystems are in series, hence $H(s) = H_1(s)H_2(s)$. With the given expression for $H_1(s)$ it follows that $H_2(s) = H_1(s) = s/(s+10)$ for $\mathrm{Re}(s) > -10$.

d) The Laplace transform of $x(t)$ is given by

$$X(s) = \frac{3}{s+5} - \frac{3}{s-5} \qquad |\mathrm{Re}(s)| < 5,$$

so that

$$Y(s) = H_1(s)X(s) = \frac{s}{s+10}\left(\frac{3}{s+5} - \frac{3}{s-5}\right) = \frac{4}{s+10} - \frac{3}{s+5} - \frac{1}{s-5}$$

for $|\mathrm{Re}(s)| < 5$. The inverse Laplace transform gives

$$y(t) = 4e^{-10t}u(t) - 3e^{-5t}u(t) - e^{5t}u(-t).$$

Question 3 (11 points)

The complex Fourier series expansion of a periodic signal x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk2\pi t}, \quad t \in \mathbb{R}, \qquad \text{where} \qquad X_k = \frac{2(-1)^k}{\pi(1-4k^2)}, \quad k \in \mathbb{Z}.$$

a) What is the fundamental period T_0 of x.

b) What is the average value (DC component) of the signal?

c) What is the energy of one period of the signal?

d) By inspection of the Fourier coefficients, we conclude that the coefficients are both symmetric ($X_k = X_{-k}$) and real ($X_k = X_k^*$), where the superscript * denotes complex conjugation. Show that this implies that the signal x is symmetric and real. That is, $x(t) = x(-t)$ and $x(t) = x^*(t)$.

e) Given the Fourier series given above, what can you say about x in terms of continuity and differentiability? (Motivate your answer.)

f) Indicate how the series converges to x (pointwise, in norm, etc). Motivate your answer.

The signal x is input to two LTI systems of which the input-output relations are given by

$$y_1(t) = 2\frac{dx(t)}{dt}, \qquad y_2(t) = x(t-1)e^{-j2\pi t}.$$

g) Compute the Fourier series of the signals y_1 and y_2 and sketch the magnitude and phase spectrum as a function of the (angular) frequency Ω.

Solution

a) $\Omega_0 = 2\pi$ so that $T_0 = \dfrac{2\pi}{\Omega_0} = 1$.

b) The DC-component is given by $X_0 = \dfrac{2}{\pi}$.

c) Parseval: $\|x\|^2 = \int_{t_0}^{t_0+1} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X_k|^2 = \sum_{k=-\infty}^{\infty} \left| \frac{2}{\pi(1-4k^2)} \right|^2$

d) Since the Fourier coefficients X_ks are real and symmetric, we can express the series as

$$\frac{2}{\pi} + \sum_{k=1}^{\infty} \frac{4(-1)^k}{\pi(1-4k^2)} \cos(k2\pi t), \quad t \in \mathbb{R},$$

which shows that x is real and symmetric.

e) The decay of the spectrum is $\mathcal{O}(1/k^2)$ from which we conclude that $x \in C^0$ (continuous but not differentiable).

f) Since x is continuous and satisfies the Dirichlet conditions, the series converges pointwise (and thus in norm).

g) $Y_{1_k} = jk4\pi X_k$ and $Y_{2_k} = e^{-j(k+1)2\pi} X_{k+1} = X_{k+1}$.
In the case of system 1, the spectrum is modified by $jk4\pi = k4\pi e^{j\pi/2}$. Hence, the magnitude spectrum of X is multiplied by $k4\pi$ and the phase spectrum gets an additional term $\pi/2$. In the case of system 2, the spectrum is shifted by one postion (2π radians)

Question 4 (11 points)

Consider a causal LTI system of which the transfer function is given by

$$H(s) = \frac{s^2}{s^2 + 2s + 5}, \quad s \in \text{ROC}$$

a) Determine the poles and zeros of the transfer function and draw them in the complex s-plane.

b) Specify the region of convergence (ROC).

c) Is the system BIBO stable? Motivate your answer.

d) Give an expression for the frequency, magnitude and phase response of the system.

e) Sketch, based on the poles and zeros of $H(s)$, the magnitude and phase response of the system and give the values of $|H(j\Omega)|$ and $\angle H(j\Omega)$ for the (angular) frequencies $\Omega = 0, \Omega = \pm\sqrt{5}$ and $\Omega = \pm\infty$.

Consider the following input signal for the above mentioned LTI system

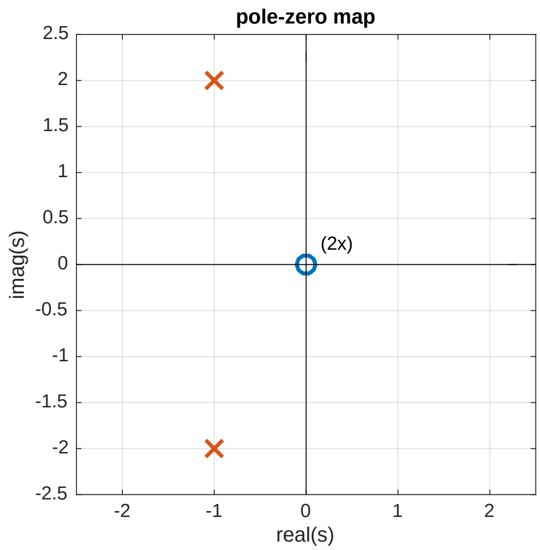
$$x(t) = \left(3 + \cos(\sqrt{5}t)\right) u(t).$$

f) Give an expression for the steady-state response of the system.

g) The LTI system described above is a high-pass filter. How can we modify the filter (by moving/adding/removing poles and zeros) such that the resulting filter has a low-pass characteristic with approximately the same cut-off frequency? Motivate your answer.

Solution

a) Double zero at $s = 0$. Poles at $s = -1 \pm 2j$.



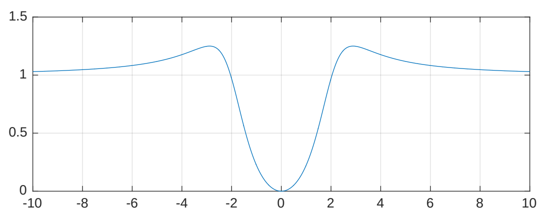
b) Since the system is causal, the ROC includes $s = +\infty$. The ROC does not include the poles and only the real part of s is of importance. Hence the ROC is $\text{Real}(s) > -1$.

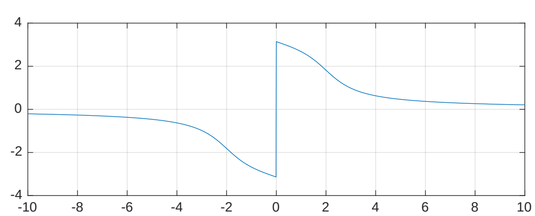
c) Since the system is causal, all poles must lie in the left half-plane. Hence, the system is BIBO stable.

d) $H(j\Omega) = \dfrac{-\Omega^2}{5 - \Omega^2 + 2j\Omega}$.

$$|H(j\Omega)| = \frac{\Omega^2}{\sqrt{(5-\Omega^2)^2 + 4\Omega^2}}, \quad \angle H(j\Omega) = \pi - \tan^{-1}\left(\frac{2\Omega}{5-\Omega^2}\right)$$

e) $|H(j0)| = \pi, |H(\pm j\sqrt{5})| = \frac{1}{2}\sqrt{5}, |H(\pm j\infty)| = 1, \angle H(j0^+) = -\angle H(j0^-) = \pi, \angle H(j\sqrt{5}) = -\angle H(-j\sqrt{5}) = \pi/2, \angle H(j\infty) = -\angle H(-j\infty) = 0$





f) $y_{\text{ss}}(t) = 3H(j0) + |H(j\sqrt{5})| \cos(\sqrt{5}t + \angle H(j\sqrt{5})) = \dfrac{1}{2}\sqrt{5}\cos(\sqrt{5}t + \pi/2)$

g) By removing the zeros and leaving the poles at their current position. Leaving out at least one zero will make the frequency response decay to zero as $\Omega \rightarrow \pm\infty$. In order to realise a low-pass characterisitc, all zeros at the origin should be removed.