Delft University of Technology
Faculty of Electrical Engineering, Mathematics and Computer Science

Partial exam EE2S11 Signals and Systems
Part 2: 31 January 2020, 13:30–15:30

Closed book; two sides of handwritten notes permitted

This exam consists of five questions (36 points)

Question 1 (11 points)

a) Determine the Fourier transform of

$$x(t) = e^{-a|t|}\cos(\Omega_0 t)$$

for $a > 0$.

b) Determine the z-transform of

$$h[n] = (\delta[n] + \delta[n-1]) * a^n u[n]\,,$$

for $|a| < 1$, where '$*$' denotes convolution. Also specify the ROC.

c) The transfer function of an FIR filter is $H(z) = z^{-2}(0.5z + 1 - 0.5z^{-1})$. Find the frequency response of this filter. Is this a linear-phase filter? (Motivate)

d) Determine the inverse z-transform of

$$H(z) = \frac{z+1}{z^2+0.81}\,, \qquad \text{ROC: } |z| > 0.9\,.$$

Is this a stable filter (why)?

e) Determine the inverse z-transform of

$$H(z) = \frac{z+1}{z^2+0.81}\,, \qquad \text{ROC: } |z| < 0.9\,.$$

Is this a stable filter (why)?

Solution

a) Using tables, and the multiplication property,

$$H(\Omega) = \frac{1}{2}\left(\frac{2a}{a^2+\Omega^2}\right) * \big(\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)\big) = a\left(\frac{1}{a^2+(\Omega-\Omega_0)^2} + \frac{1}{a^2+(\Omega+\Omega_0)^2}\right).$$

b)

$$H(z) = (1+z^{-1})\cdot\frac{1}{1-az^{-1}} = \frac{1+z^{-1}}{1-az^{-1}}\,, \qquad \text{ROC: } |z| > a\,.$$

c)

$$H(e^{j\omega}) = e^{-2j\omega}(0.5e^{j\omega} + 1 - 0.5e^{-j\omega}) = e^{-2j\omega}(1 + j\sin(\omega))\,.$$

This is not a linear-phase filter because "1" and "$j\sin(\omega)$" do not combine into a purely real or purely imaginary term.

d) There are several approaches. The first one: write

$$H(z) = \frac{z+1}{z^2+0.81} = \frac{A}{z+0.9j} + \frac{B}{z-0.9j}$$

where we find $A = \frac{1}{2} + j\frac{5}{9}$, $B = \frac{1}{2} - j\frac{5}{9}$. The given ROC contains $z \to \infty$, hence $h[n]$ is causal. Thus, rewrite as

$$H(z) = \frac{Az^{-1}}{1+0.9jz^{-1}} + \frac{Bz^{-1}}{z-0.9jz^{-1}}$$

$$\Rightarrow \quad h[n] = A(-0.9j)^{n-1}u[n-1] + B(0.9j)^{n-1}u[n-1]\,.$$

Alternatively, first write

$$H(z) = \frac{z^{-1}+z^{-2}}{1+0.81z^{-2}} = (z^{-1}+z^{-2})\left(\frac{0.5}{1+0.9j\ z^{-1}} + \frac{0.5}{1-0.9j\ z^{-1}}\right)$$

Using a table, we find

$$\frac{0.5}{1+0.9j\ z^{-1}} + \frac{0.5}{1-0.9j\ z^{-1}} \quad \Leftrightarrow \quad 0.5\big((-0.9j)^n + (0.9j)^n\big)u[n] = (0.9)^n\cos(\frac{\pi}{2}n)u[n]$$

The delay property of z^{-1} gives

$$h[n] = (0.9)^{n-1}\cos(\frac{\pi}{2}(n-1))u[n-1] + (0.9)^{n-2}\cos(\frac{\pi}{2}(n-2))u[n-2]$$

which could be written in various other forms as well.

This is a stable filter as the unit circle is contained in the ROC.

e) This ROC contains $z = 0$ hence $x[n]$ is now anti-causal. We rewrite

$$\frac{1}{z^2+0.81} = \frac{1}{0.81}\frac{1}{1+\frac{1}{0.81}z^2} = \frac{1}{2\cdot 0.81}\left(\frac{1}{1+\frac{1}{0.9}j\ z} + \frac{1}{1-\frac{1}{0.9}j\ z}\right)$$

which transforms to

$$\frac{1}{2\cdot 0.81}\big((-\frac{1}{0.9}j)^{-n}u[-n] + (\frac{1}{0.9}j)^{-n}\big)u[-n] = \frac{1}{0.81}(0.9)^n\cos(\frac{\pi}{2}n)u[-n]$$

Altogether,

$$h[n] = \frac{1}{0.81}(0.9)^{n+1}\cos(\frac{\pi}{2}(n+1))u[-n-1] + \frac{1}{0.81}(0.9)^{n+2}\cos(\frac{\pi}{2}(n+2))u[-n-2]\,.$$

This is an unstable filter as the unit circle is not contained in the ROC.

Question 2 (8 points)

The output of a discrete-time causal filter with transfer function

$$H(z) = \frac{z+1}{z^2+0.81}$$

is a sequence $y[n] = \delta[n-1] + \delta[n-2]$.

a) Determine the input sequence $x[n]$ such that the output of the filter is the given $y[n]$.

b) Determine all poles and zeros of the filter and draw a pole-zero plot.

c) Using b), sketch the amplitude spectrum $|H(e^{j\omega})|$, also indicate values on the axes.

d) Draw the "Direct form no. II" realization of the filter and also specify the coefficients.

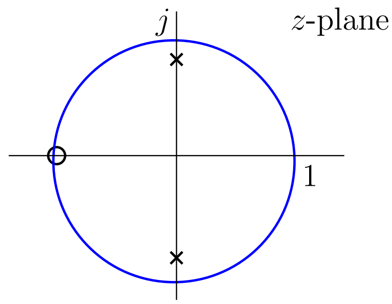
Solution

a) $Y(z) = z^{-1} + z^{-2}$, hence

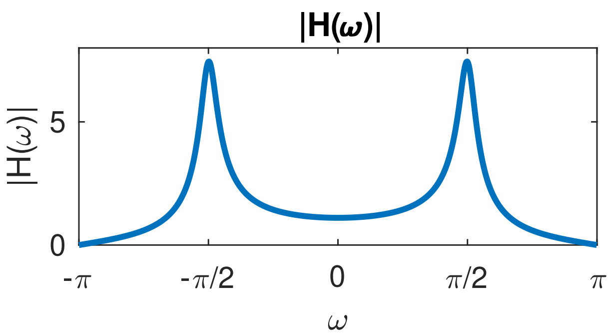
$$X(z) = \frac{Y(z)}{H(z)} = 1 + 0.81z^{-2}$$

and $x[n] = \delta[n] + 0.81\delta[n-2]$.

b) Poles: $z_p = \pm 0.9j$, zeros: $z_n = -1, \infty$.



c)

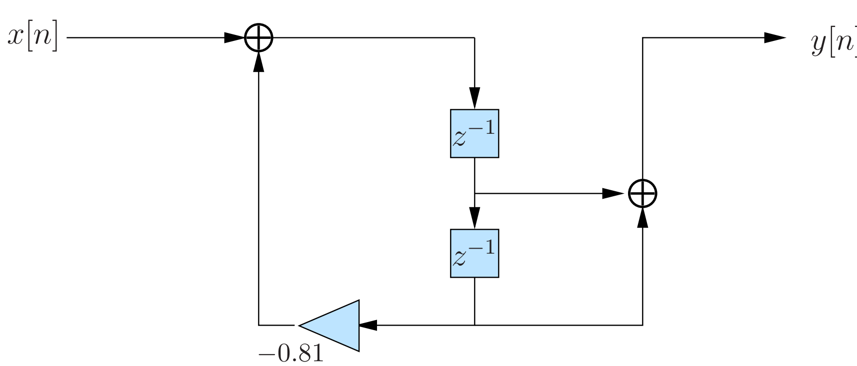


Close to poles, the response peaks. At $\omega = \pi$, the response is zero because there is a zero at $z = -1$. At $\omega = 0$ ($z = 1$), the response is not zero, but not very big either. Calculate $H(e^{j0}) = 2/1.81 = 1.11$.

The spectrum is periodic with period 2π; only one period is plotted.

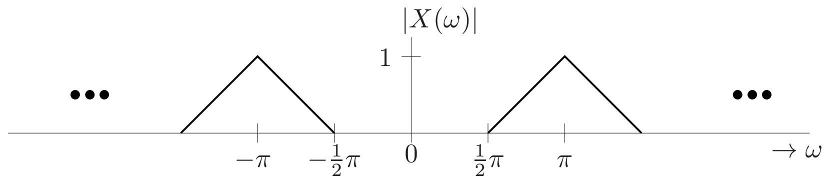
d) First write:

$$H(z) = \frac{z^{-1} + z^{-2}}{1 + 0.81z^{-2}}$$



Question 3 (6 points)

Consider an analog signal $x_a(t)$ with Fourier transform $X_a(F)$ (with F in Hz). Suppose that the signal is bandlimited with maximal frequency 5 kHz. The signal is sampled at $F_s = 10$ kHz. The resulting discrete-time signal $x[n]$ has spectrum $X(\omega)$ as shown below:



a) Using ideal components, is it possible to recover $x_a(t)$ from $x[n]$? (How? Or why not?)

b) What is the relation between F and ω?

c) Plot $|X_a(F)|$.

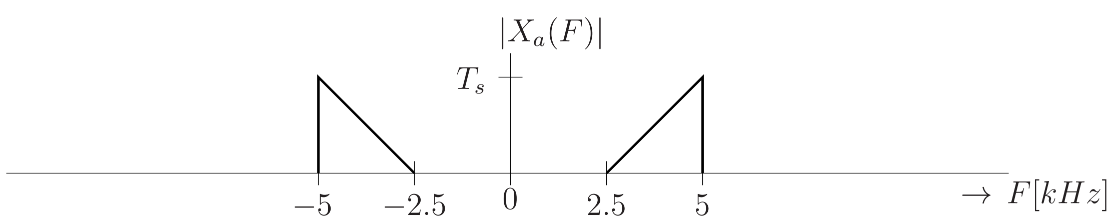
d) Suppose that, instead, we sample $x_a(t)$ at 5 kHz. Draw the spectrum of the resulting digital signal (also clearly mark the frequencies).

Solution

a) The Nyquist condition is satisfied: no aliasing, hence perfect reconstruction is possible by using an ideal DAC followed by an ideal lowpass filter (with cut-off at 5 kHz).

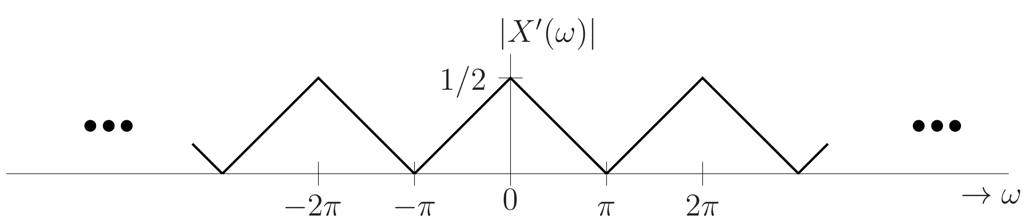
b) F_s corresponds to $\omega_s = 2\pi$, hence $\omega = 2\pi \frac{F}{F_s} = \frac{2\pi}{10{,}000} F$.

c)



Note that the spectrum is not periodic. Note it was specified that the signal is bandlimited at 5 kHz, so there are no components above 5 kHz. (This also follows from the ideal reconstruction in item a.)

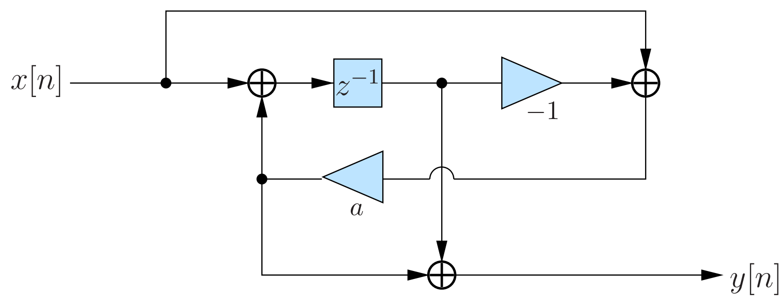
d) This time, there is aliasing.



(Note the change in vertical axis.)

Question 4 (4 points)

a) Determine the transfer function $H(z)$ of the following realization:



b) Is this a minimal realization? (Why?)

Solution

a) Call the output of the delay element $P(z)$, write down the equations

$$\begin{cases} P &= z^{-1}(X + a(X - P)) \\ Y &= P + a(X - P) \end{cases}$$

and eliminate $P(z)$; this results in

$$H(z) = \frac{a + z^{-1}}{1 + az^{-1}}$$

b) Minimal: first order, and the realization used one delay.

(While the given realization looks chaotic, its advantage is that it uses only one multiplier, and the resulting transfer function is allpass for any value of a.)

Question 5 (7 points)

A second-order analog lowpass filter is given by

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1} .$$

a) Determine the squared magnitude response, $|H(j\Omega)|^2$, and give a sketch of it. In the plot, also specify the cut-off frequency.

b) What transformation is needed to obtain a high-pass filter with cut-off frequency Ω_c [rad/s]? Give an expression for the resulting high-pass filter $G_a(s)$.

We want to use $G_a(s)$ to design a second order *digital* high-pass filter with cut-off frequency $\omega_c = 0.2\pi$ [rad].

c) Is the bilinear transform suitable? (Motivate.)

d) Give an expression for the resulting digital high-pass filter $G(z)$.

Solution

a)

$$|H(j\Omega)|^2 = H(s)H(-s)\big|_{s=j\Omega} = \frac{1}{(s^2+1)^2 - 2s^2}\Big|_{s=j\Omega} = \frac{1}{s^4+1}\Big|_{s=j\Omega} = \frac{1}{1+\Omega^4}\,.$$

This is recognized as a normalized second-order Butterworth filter. The cut-off frequency is $\Omega_c = 1$.

b) Low-pass to high-pass:

$$s \to \frac{\Omega_c}{s}\,, \qquad \Omega \to \frac{\Omega_c}{\Omega}\,.$$

$$G_a(s) = \frac{1}{\left(\frac{\Omega_c}{s}\right)^2 + \sqrt{2}\left(\frac{\Omega_c}{s}\right) + 1} = \frac{s^2}{s^2 + \sqrt{2}\Omega_c s + \Omega_c^2}\,.$$

c) Bilinear transform is suitable, also for highpass filters, as it does not lead to aliasing but just stretches the frequency axis (i.e., a one-to-one mapping).

d) The bilinear transform is

$$s \to \frac{1-z^{-1}}{1+z^{-1}}\,, \qquad \Omega_c = \tan(\frac{\omega_c}{2})$$

Substitution leads to $\Omega_c = 0.3249$ and

$$G(z) = \frac{(1-z^{-1})^2}{(1-z^{-1})^2 + \sqrt{2}\Omega_c(1-z^{-2}) + \Omega_c^2(1+z^{-1})^2}$$

which could be rearranged a bit further.