Chapter 9

Discrete–time Signals and Systems

9.1 Basic Problems

9.1 See Fig. 9.1 for some of the answers.
Expressing $x[n] = \delta[n+1] + \delta[n] + \delta[n-1] + 0.5\delta[n-2]$ we then have
(a) $x[n-1]$ is $x[n]$ delayed by 1 (shifted right 1 sample); $x[-n]$ is the reflection of $x[n]$, and $x[-n+2]$ is $x[n]$ reflected and shifted right by 2 samples, or $x[-n+2] = 0.5\delta[-n] + \delta[-n+1] + \delta[-n+2] + \delta[-n+3] = 0.5\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]$ because $\delta[n]$ is even.
(b)(c) Even

$$x_e[n] = 0.5(x[n] + x[-n]) \begin{cases} 1 & n = -1, 0, 1 \\ 0.25 & n = -2, 2 \\ 0 & \text{otherwise} \end{cases}$$

and the odd component is

$$x_o[n] = 0.5(x[n] - x[-n]) \begin{cases} -0.25 & n = -2 \\ 0.25 & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

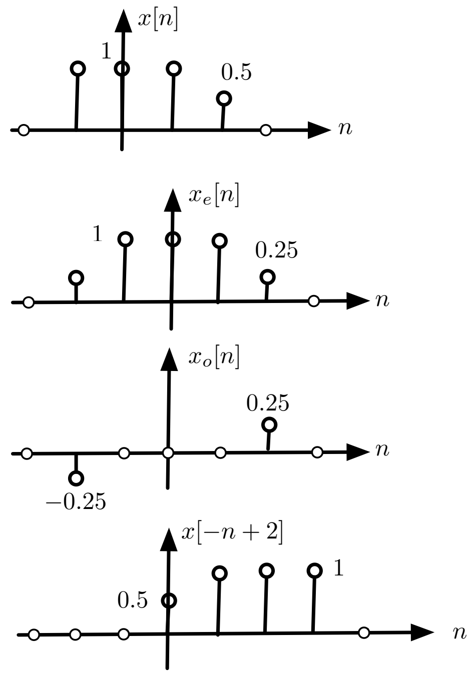

Figure 9.1: Problem 1

9.2 (a) The discrete frequency of $x[n]$ is

$$\omega_0 = 0.7\pi = \frac{2\pi \times 7}{20}$$

so that the fundamental period is $N_0 = 20$ and $m = 7$.

(b) $x(t)|_{t=nT_s} = x(nT_s) = \cos(\pi n T_s)$, so $T_s = 0.7$ sec/sample, and the sampling frequency $\Omega_s = 2\pi/T_s = \pi \times (20/7)$ rad/sec which satisfies Nyquist since $\Omega_{max} = \pi$ rad/sec. i.e., $\Omega_s > 2\Omega_{max}$. The discrete–time signal here coincides with the one given above.

(c) To make $\cos(\pi n T_s)$ look like $\cos(\pi t)$ we need to divide its fundamental period into large number of points, e.g., the fundamental period of $x(t)$ is $T_0 = 2$, letting $T_s = T_0/N$ for $N \geq 2$ would satisfy Nyquist but also would provide a discrete signal that looks like the analog signal when N is large. For instance, in this case $T_s \leq 1$ so that $T_s = 1$ ($N = 2$) and $T_s = 0.1$ ($N = 20$) both satisfy Nyquist but the latter one gives a signal that looks like the analog signal while the other does not.

9.3 (a) The discrete frequency for the given signals are

$$\begin{array}{lll}(i) & x[n]: & \omega_0 = \pi = \dfrac{2\pi}{2} \;\; \Rightarrow \;\; \text{periodic with period } \; N_0 = 2\\ (ii) & y[n]: & \omega_0 = 1 \neq \dfrac{2\pi m}{N_0}, \text{ not periodic}\\ (iii) & z[n]: & \text{ not periodic, as } y[n] \text{ is not periodic}\\ (iv) & v[n]: & \omega_0 = \dfrac{3\pi}{2} = \dfrac{2\pi}{4}3, \;\; \Rightarrow \;\; \text{periodic with period } \; N_0 = 4\end{array}$$

(b) $x_1[n]$ is periodic of fundamental period $N_1 = 4$, and $y_1[n]$ is periodic of fundamental period $N_2 = 6$ so that

$$\frac{N_1}{N_2} = \frac{4}{6} = \frac{2}{3}$$

then the sum $z_1[n] = x_1[n] + y_1[n]$ is periodic of period $3N_1 = 2N_2 = 12$, i.e., three periods of $x_1[n]$ fit in 2 of $y_1[n]$.
Similarly, $v_1[n]$ is periodic of period 12. Indeed,

$$v_1[n+12] = x_1[n+12]y_1[n+12] = x_1[n]y_1[n]$$

since 12 is three times the period of $x_1[n]$ and two times the period of $y_1[n]$.
The compressed signal $w_1[n] = x_1[2n]$ has period $N_1/2 = 2$:

$$w_1[n+2] = x_1[2(n+2)] = x_1[2n+4] = x_1[2n] = w_1[n]$$

since $x_1[2n]$ is periodic of period 2.

9.6 (a) i. If input is $x[n]$ the output is $y[n] = x[n]x[n-1]$, if input is $\alpha x[n]$, $\alpha \neq 1$, the output is $\alpha^2 x[n]x[n-1] \neq \alpha x[n]x[n-1]$ so system is non-linear.
If input is $x[n]$ the output is $y[n] = x[n]x[n-1]$, and if input is $x[n-1]$ the output is $x[n-1]x[n-2] = y[n-1]$ so system is time-invariant.

ii. Causal, $y[n]$ depends on present and past inputs and it is zero when the input is zero. Moreover, when $x[n] = 0$ then $y[n] = 0$.
If $x[n]$ is bounded for all n, then $|x[n]| < M$ and $|x[n-1]| < M$ and $|y[n]| = |x[n]||x[n-1]| < M^2$, so bounded. Yes, BIBO stable.
Indeed, if $x[n] = u[n]$ then
Non-linearity:

$$\begin{aligned} x[n] &= u[n] \;\Rightarrow\; y[n] = u[n]u[n-1] = u[n-1] \\ x_1[n] &= 2u[n] \;\Rightarrow\; y_1[n] = 4u[n-1] \neq 2u[n-1] \end{aligned}$$

Time-invariance

$$\begin{aligned} x[n] = u[n] \quad &\Rightarrow \quad y[n] = u[n-1] \\ x_1[n] = u[n-1] \quad &\Rightarrow \quad y_1[n] = u[n-2] = y[n-1] \end{aligned}$$

(b) i. No, it is a modulation system as such LTV.

ii. Expressing $x[n] = \cos(2\pi n/8)$ its fundamental period is $N_0 = 8$. We have

$$y[n+8] = x[n+8]\cos((n+8)/4) = x[n]\cos(n/4+2) \neq y[n]$$

since 2 cannot be expressed in terms of π, so $y[n]$ is not periodic. It will also be true for any multiple of 8.

9.9 (a) If input is $\alpha x[n]$, output is $\sum_{k=n-2}^{n+4} \alpha x[k] = \alpha y[n]$, so system is linear. If input is $x_1[n] = x[n-1]$ the output is

$$\begin{aligned} y_1[n] &= \sum_{k=n-2}^{n+4} x_1[k] = \sum_{k=n-2}^{n+4} x[k-1] \;\; \text{let } m = k-1 \\ &= \sum_{m=n-3}^{n+3} x[m] = y[n-1] \end{aligned}$$

so the system is time–invariant.

(b) Non–causal since output depends on future values of input:

$$y[n] = \sum_{k=n-2}^{n+4} x[k] = \sum_{k=n-2}^{n} x[k] + \sum_{k=n+1}^{n+4} x[k]$$

If $|x[n]| < M$, i.e., bounded, then $y[n]$ is bounded, indeed

$$|y[n]| \leq \sum_{k=n-2}^{n+4} |x[k]| \leq \sum_{k=n-2}^{n+4} M = 7M$$

9.10 (a) Impulse response: using the difference equation

$$h[n] = -0.5h[n-1] + \delta[n]$$

obtained when the input is $\delta[n]$ and the outputs is $h[n]$ (no initial conditions) we have

$$\begin{aligned} h[n] &= -0.5h[n-1] + \delta[n] \\ &= -0.5(-0.5h[n-2] + \delta[n-1]) + \delta[n] = 0.5^2 h[n-2] - 0.5\delta[n-1] + \delta[n] \\ &= 0.5^2(-0.5h[n-3] + \delta[n-2]) - 0.5\delta[n-1] + \delta[n] = -0.5^3 h[n-3] \\ &\quad +0.5^2\delta[n-2]) - 0.5\delta[n-1] + \delta[n] \\ &\quad \vdots \end{aligned}$$

indicating that $h[n] = (-0.5)^n u[n]$.

(b) Writing $x[n] = \delta[n] + \delta[n-1] + \delta[n-2]$ the output is

$$\begin{aligned} y[n] &= h[n] + h[n-1] + h[n-2] \\ &= (-0.5)^n u[n] + (-0.5)^{n-1} nu[n-1] + (-0.5)^{n-2} u[n-2] \end{aligned}$$

Recursive solution of the difference equation gives the same result.

9.11 (a) The convolution integral gives

$$y(t) = r(t) - r(t-2.5) - r(t-3.5) + r(t-6)$$

See Fig. 9.2

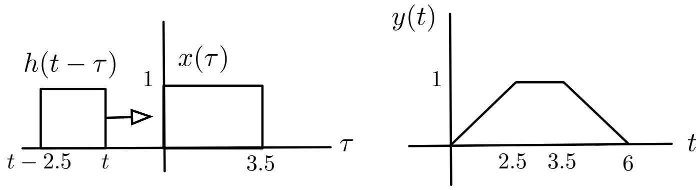

Figure 9.2: Problem 11(a)

(b) If we discretize both $x[n]$ and $h[n]$ with $T_s = 0.5$, i.e., $t = 0.5n$, we get

$$x[n] = \begin{cases} 1 & 0 \leq \frac{n}{2} \leq 3.5 \text{ or } 0 \leq n \leq 7 \\ 0 & \text{otherwise} \end{cases}$$

$$h[n] = \begin{cases} 1 & 0 \leq \frac{n}{2} \leq 2.5 \text{ or } 0 \leq n \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

The convolution sum gives

$$y[n] = \begin{cases} n+1 & 0 \leq n \leq 5 \\ 6 & n = 6,\ 7 \\ 6-(n-7) & 8 \leq n \leq 12 \\ 0 & \text{otherwise} \end{cases}$$

Thus $y[n]/6$ approximates the continuous convolution. Notice that the length of the convolution is length $x[n]+$ length of $h[n] = 8 + 6 - 1 = 13$.

9.13 Discretization of differential equation:

$$\frac{y(nT+T)-y(nT)}{T} + y(nT) = 2x(nT) + \frac{x(nT+T)-x(nT)}{T}$$
$$\Rightarrow \quad y((n+1)T) = (1-T)y(nT) + (2T-1)x(nT) + x((n+1)T)$$

and letting $m = n+1$ we get

$$y(mT) = (1-T)y((m-1)T) + (2T-1)x((m-1)T) + x(mT)$$

Figure 9.3 shows the block diagram when $T = 1$.

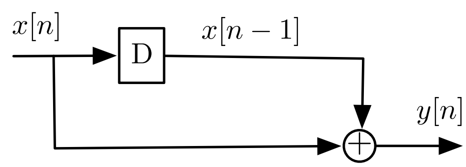

Figure 9.3: Problem 13: block diagram of difference equation when $T = 1$.

9.14 (a) We have

$$e[n] = x[n] - y[n-1], \;\; y[n] = 2e[n-1] = 2x[n-1] - 2y[n-2]$$
$$\Rightarrow \quad y[n] + 2y[n-2] = 2x[n-1]$$

(b) The impulse response is found by letting $x[n] = \delta[n]$, IC zero, and $y[n] = h[n]$. Recursively,

$$h[n] = -2h[n-2] + 2\delta[n-1]$$
$$h[0] = 0, \;\; h[1] = 2, \;\; h[2] = 0, \;\; h[3] = -4, \;\; h[4] = 0, \;\; h[5] = 8, \cdots$$

or $h[n] = 2(-2)^{(n-1)/2}$, n odd, 0 otherwise, which grows as n increases, so it is not absolutely summable, i.e., system is not BIBO stable.

9.15 (a) $x[n] = \delta[n] + \delta[n-1] + \delta[n-2]$ and $y[n] = \delta[n-1] + \delta[n-2] + \delta[n-3]$, and

$$\text{length }(y[n]) = \text{length }(x[n]) + \text{length }(h[n]) - 1.$$

Thus length of $h[n]$ is $3 - 3 + 1 = 1$.

(b) Since the input is $x[n] = \delta[n] + \delta[n-1] + \delta[n-2]$ the output is $y[n] = h[n] + h[n-1] + h[n-2]$ but also $y[n] = \delta[n-1] + \delta[n-2] + \delta[n-3]$ so that

$$\begin{aligned} y[0] &= 0 = h[0] \\ y[1] &= 1 = h[1] + h[0] \\ y[2] &= 1 = h[2] + h[1] + h[0] \\ y[3] &= 1 = h[3] + h[2] + h[1] \end{aligned}$$

solving for the impulse response values we get $h[0] = 0$, $h[1] = 1 - h[0] = 1$ and the rest of the values are zero. Thus the length of $h[n]$ is 1.

9.17 (a) For $N = 4$ the length of the output $y[n]$ = length of $x[n]$+ length $h[n] - 1 = 5 + 4 - 1 = 8$. For $N = 4$, the convolution sum gives (see Fig. 9.4):

n	$y[n]$	n	$y[n]$
$0,\ 7$	1	$2,\ 5$	3
$1,\ 6$	2	$3,\ 4$	4

0 otherwise.

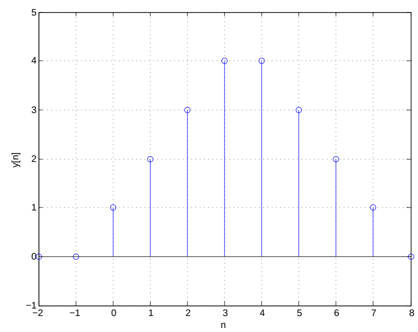

Figure 9.4: Problem 17: convolution sum $y[n]$ of $x[n]$ and $h[n]$

(b) The values of the convolution sum

$$
\begin{aligned}
y[3] &= \sum_{k=0}^{3} h[k]x[3-k] = x[3] + x[2] + x[1] + x[0] = 3 \\
y[6] &= \sum_{k=0}^{6} h[k]x[6-k] = x[6] + x[5] + x[4] + x[3] = 0
\end{aligned}
$$

can be obtained by letting $x[n] = u[n] - u[n-3]$ so that the second summation is zero as $x[n] = 0$ for $n \geq 3$ and $h[n] = 0$ for $n \geq 4$, and the first summation is 3 given that $x[3] = 0$ and the other terms are one.

9.18 (a) Solving recursively the first difference equation $y[n] = 0.5y[n-1] + x[n]$, with $x[n] = \delta[n]$, IC zero:

$$\begin{aligned} y[0] &= \delta[0] = 1 \\ y[1] &= 0.5 \times 1 + 0 = 0.5 \\ &\vdots \\ y[n] &= 0.5^n \end{aligned}$$

For the second difference equation $y[n] = 0.25y[n-2] + 0.5x[n-1] + x[n]$, with $x[n] = \delta[n]$, IC zero:

$$\begin{aligned} y[0] &= 0 + 0 + \delta[0] = 1 \\ y[1] &= 0 + 0.5\delta[0] + 0 = 0.5 \\ y[2] &= 0.25 + 0 + 0 = 0.5^2 \\ &\vdots \\ y[n] &= 0.5^n \end{aligned}$$

The second equation is obtained by replacing $y[n-1] = 0.5y[n-2] + x[n-1]$ (calculated by changing n by $n-1$ in the first difference equation) into the first equation.

(b) Replacing $y[n-1] = 0.5y[n-2] + x[n-1]$ we get the previous second equation, then replacing $y[n-2] = 0.5y[n-3] + x[n-2]$ we get a new equation,

$$y[n] = 0.5^3 y[n-3] + 0.5^2 x[n-2] + 0.5x[n-1] + x[n]$$

and repeating this process we finally obtain

$$y[n] = \sum_{k=0}^{\infty} 0.5^k x[n-k]$$

which is the convolution sum of $h[n] = 0.5^n u[n]$ (impulse response) and $x[n]$ which coincides with the response obtained above.

(c) For $x[n] = u[n] - u[n-11]$ the convolution sum is (do the convolution sum graphically to verify these results)

$$y[n] = \begin{cases} 0 & n < 0 \\ \sum_{k=0}^{n} 0.5^k & 0 \leq n \leq 10 \\ \sum_{k=n-10}^{n} 0.5^k & n \geq 11 \end{cases}$$

For $0 \leq n \leq 10$ we get

$$y[n] = \frac{1 - 0.5^{n+1}}{1 - 0.5} = 2(1 - 0.5^{n+1})$$

for $n \geq 11$, letting $m = k - n + 10$

$$y[n] = \sum_{m=0}^{10} 0.5^{m+n-10} = 0.5^{n-10} \times 2(1 - 0.5^{11})$$

As $n \to \infty$ we get $y[n] \to 0$.

(d) The maximum occurs at $n = 10$ when $y[10] = 2(1 - 0.5^{11})$.

9.19 The convolution sum is

$$
\begin{aligned}
y[n] &= \sum_{m=0}^{n} x[m]h[n-m] \\
y[0] &= x[0]h[0] = 1 \\
y[1] &= x[0]h[1] + x[1]h[0] = -1 + 1 = 0 \\
y[2] &= x[0]h[2] + x[1]h[1] + x[2]h[0] = 1 - 1 + 1 = 1 \\
y[3] &= x[0]h[3] + x[1]h[2] + x[2]h[1] + x[3]h[0] = -1 + 1 - 1 + 0 = -1 \\
y[4] &= x[0]h[4] + x[1]h[3] + x[2]h[2] + x[3]h[1] + x[4]h[0] = 1 - 1 + 1 + 0 + 0 = 1 \\
y[n] &= x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] = (-1)^n + (-1)^{n-1} + (-1)^{n-2} = (-1)^{n-2} \;\; n \geq 5
\end{aligned}
$$

9.25 (a) $y[n]$ depends on a future value of the input $x[n+1]$ so the system is non-causal.

(b) This can be done in two equivalent ways:
(i) If $x[n]$ is bounded, i.e., there is a value M such that $|x[n]| < M < \infty$, then

$$|y[n]| \leq \frac{1}{3}\left(|x[n+1]| + |x[n]| + |x[n-1]|\right) \leq \frac{3M}{3} = M \leq \infty$$

or bounded, so that the system is BIBO stable.
(ii) System is BIBO stable if its impulse response is absolutely summable. If $x[n] = \delta[n]$ then

$$h[n] = \frac{1}{3}\left(\delta[n+1] + \delta[n] + \delta[n-1]\right)$$

and

$$\sum_n |h[n]| = |h[-1]| + |h[0]| + |h[1]| = 1 < \infty$$

so the system is BIBO stable.

(c) The discrete-time signal is

$$x[n] = 2\cos(10t)|_{t=nT_{si}} = \begin{cases} 2\cos(10n) & \text{when using } T_{s1} = 1 \\ 2\cos(10\pi n) & \text{when using } T_{s2} = \pi \end{cases}$$

For $T_{s1} = 1$, the frequency of $x[n]$ is $\omega_0 = 10$ which cannot be expressed as $2\pi m/N$ for not canceling integers m and N, so $x[n]$ is then not periodic.
For $T_{s2} = \pi$, the frequency of $x[n]$ is

$$\omega_0 = 10\pi = \frac{2\pi m}{N} = \frac{2\pi \times 5}{1}$$

so that $x[n]$ in that case is periodic of period $N = 1$.

(d) $y[n]$ is periodic as $x[n]$, $x[n+1]$ and $x[n-1]$ are periodic of same period. The fundamental period of $y[n]$ is then $N = 1$.

9.27 (a) If $x[n] = \delta[n]$ then $h[n] = \delta[n] - \delta[n-5]$. System is causal and BIBO stable.

(b) $x[n] = u[n] = \cos(0n)u[n]$ has infinite energy, and the corresponding output is $y[n] = u[n] - u[n-5]$ having finite energy. Notice the output has a finite support, as when $n > 5$ then $y[n] = 0$.

(c) When $x[n] = \sin(2\pi n/5)u[n]$, then the output is

$$\begin{aligned} y[n] &= x[n] - x[n-5] = \sin(2\pi n/5)u[n] - \sin(2\pi(n-5)/5)u[n-5] \\ &= \sin(2\pi n/5)(u[n] - u[n-5]) \end{aligned}$$

which again has finite support and $y[n] = 0$ for $n > 5$. The energy of the output is finite, while that of the input is not.

(d) Since $x[n] = (e^{j\omega_0 n} - e^{-j\omega_0 n})u[n]/2j$ and the system is causal, linear and time invariant, consider the response to $e^{j\omega_0 n}u[n]$. The convolution sum, or response to $x_1[n] = e^{j\omega_0 n}u[n]/2j$

$$y_1[n] = \sum_{k=0}^{n} h[k]e^{j\omega_0(n-k)}/(2j)$$

given that $h[n] = \delta[n] - \delta[n-5]$, or $h[0] = 1$, $h[5] = -1$ and the other values are zero, for $n \geq 5$ we have

$$\begin{aligned} y_1[n] &= \sum_{k=0}^{n} h[k]e^{j\omega_0(n-k)}/(2j) = e^{j\omega_0 n}(h[0] + h[5]e^{-j5\omega_0})/(2j) \\ &= e^{j\omega_0 n}(1 - e^{-j5\omega_0})/(2j) \end{aligned}$$

which can be made zero by letting $1 - e^{-j5\omega_0} = 0$ or for frequencies $\omega_0 = 2\pi m/5$ for $m = 0, \pm 1, \pm 2, \cdots$. Similarly when the input is $e^{-j\omega_0 n}u[n]$. Thus for those frequencies the output is of finite support, i.e., having finite energy. For any other frequencies the output is not zero after $n \geq 5$, and it is not guaranteed the finite support or the finite energy.