Chapter 5

Frequency Analysis: the Fourier Transform

5.1 Basic Problems

5.1 (a) The Laplace transforms are

$$x_1(t) = e^{-2t}u(t) \quad \Leftrightarrow \quad X_1(s) = \frac{1}{s+2} \qquad \sigma > -2$$
$$x_2(t) = r(t) \quad \Leftrightarrow \quad X_2(s) = \frac{1}{s^2} \qquad \sigma > 0$$
$$x_3(t) = te^{-2t}u(t) \quad \Leftrightarrow \quad X_3(s) = \frac{1}{(s+2)^2} \qquad \sigma > -2$$

(b) The Laplace transforms of $x_1(t)$ and of $x_3(t)$ have regions of convergence containing the $j\Omega$-axis, and so we can find their Fourier transforms from their Laplace transforms by letting $s = j\Omega$
(c) The Fourier transforms of $x_1(t)$ and $x_3(t)$ are

$$X_1(\Omega) = \frac{1}{2+j\Omega}$$
$$X_3(\Omega) = \frac{1}{(2+j\Omega)^2}$$

5.2 (a) In this case we are using the duality of the Fourier transforms so that the Fourier transform of the sinc is a pulse of magnitude A and cut-off frequency Ω_0 which we will need to determine.

The inverse Fourier transform is

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} A[u(\Omega+\Omega_0)-u(\Omega-\Omega_0)]e^{j\Omega t}d\Omega \\ &= \frac{A}{2\pi}\int_{-\Omega_0}^{\Omega_0} e^{j\Omega t}d\Omega \\ &= \frac{A}{\pi t}\sin\Omega_0 t \end{aligned}$$

so that $A=\pi$ and $\Omega_0=1$, i.e.,

$$\frac{\sin(t)}{t} \quad \Leftrightarrow \quad \pi[u(\Omega+1)-u(\Omega-1)]$$

(b) The Fourier transform of $x_1(t)=u(t+0.5)-u(t-0.5)$ is

$$X_1(\Omega)=\left[\frac{1}{s}[e^{0.5s}-e^{-0.5s}]\right]_{s=j\Omega}=\frac{\sin(0.5\Omega)}{0.5\Omega}$$

Using the duality property we have:

$$\begin{aligned} x_1(t)=u(t+0.5)-u(t-0.5) \quad &\Leftrightarrow \quad X_1(\Omega)=\frac{\sin(\Omega/2)}{\Omega/2} \\ X_1(t)=\frac{\sin(t/2)}{t/2} \quad &\Leftrightarrow \quad 2\pi[u(\Omega+0.5)-u(\Omega-0.5)] \end{aligned}$$

using the fact that $x_1(t)$ is even. Then using the scaling property

$$\begin{aligned} X_1(2t)=\frac{\sin(t)}{t} \quad &\Leftrightarrow \quad \frac{2\pi}{2}[u((\Omega/2)+0.5)-u((\Omega/2)-0.5)] \\ &\Leftrightarrow \quad \pi[u(\Omega+1)-u(\Omega-1)] \end{aligned}$$

so $x(t)=X_1(2t)=\sin(t)/t$ is the inverse Fourier transform of $X(\Omega)=\pi[u(\Omega+1)-u(\Omega-1)]$

5.3 (a) The signal $x(t)$ is even while $y(t)$ is odd.
(b) The Fourier transform of $x(t)$ is

$$\begin{aligned} X(\Omega) &= \int_{-\infty}^{\infty} e^{-|t|} e^{-j\Omega t} dt \\ &= \int_{-\infty}^{\infty} e^{-|t|} \cos(\Omega t) dt - j \int_{-\infty}^{\infty} e^{-|t|} \sin(\Omega t) dt \\ &= 2 \int_{0}^{\infty} e^{-|t|} \cos(\Omega t) dt \end{aligned}$$

this is because the imaginary part is the integral of an odd function which is zero. Since $\cos(.)$ is an even function

$$X(-\Omega) = X(\Omega)$$

The Fourier transform $X(\Omega)$ is

$$\begin{aligned} X(\Omega) &= 2 \int_{0}^{\infty} e^{-t} \frac{e^{j\Omega t} + e^{-j\Omega t}}{2} dt \\ &= \int_{0}^{\infty} e^{-(1-j\Omega)t} dt + \int_{0}^{\infty} e^{-(1+j\Omega)t} dt \\ &= \frac{1}{1-j\Omega} + \frac{1}{1+j\Omega} = \frac{2}{1+\Omega^2} \end{aligned}$$

which is real-valued.
(c) For $y(t)$, odd function, its Fourier transform is

$$\begin{aligned} Y(\Omega) &= \int_{-\infty}^{\infty} y(t) e^{-j\Omega t} dt \\ &= -j \int_{-\infty}^{\infty} y(t) \sin(\Omega t) dt \end{aligned}$$

because $y(t)\cos(\Omega t)$ is an odd function and its integral is zero. The $Y(\Omega)$ is odd since

$$\begin{aligned} Y(-\Omega) &= -j \int_{-\infty}^{\infty} y(t) \sin(-\Omega t) dt \\ &= -Y(\Omega) \end{aligned}$$

since the sine is odd.
(d) Let's use the Laplace transform to find the Fourier transform of $y(t)$:

$$Y(s) = \frac{1}{s+1} - \frac{1}{-s+1}$$

with a region of convergence $-1 < \sigma < 1$, which contains the $j\Omega$-axis. So

$$Y(\Omega) = Y(s)\,|_{s=j\Omega} = \frac{1}{j\Omega+1} - \frac{1}{-j\Omega+1} = \frac{-2j\Omega}{1+\Omega^2}$$

which as expected is purely imaginary.
<u>Check:</u> Let $z(t) = x(t) + y(t) = 2e^{-t}u(t)$ which has a Fourier transform

$$Z(\Omega) = \frac{2}{1 + j\Omega} = \frac{2(1 - j\Omega)}{1 + \Omega^2} = X(\Omega) + Y(\Omega)$$

(f) If a signal is represented as $x(t) = x_e(t) + x_o(t)$ then

$$X(\Omega) = X_e(\Omega) + X_o(\Omega)$$

where the first is a cosine transform and the second a sine transform.

5.5 (a) $x_1(t) = -x(t+1) + x(t-1)$, time–shift property

$$X_1(\Omega) = X(\Omega)(-e^{j\Omega} + e^{-j\Omega}) = -2jX(\Omega)\sin(\Omega)$$

(b) $x_2(t) = 2\sin(t)/t$ by duality

$$X_2(\Omega) = 2\pi[u(-\Omega+1) - u(-\Omega-1)] = 2\pi[u(\Omega+1) - u(\Omega-1)]$$

by symmetry of $x(t)$.

(c) Compression

$$x_3(t) = 2x(2t) = 2[u(2t+1) - u(2t-1)] = 2[u(t+0.5) - u(t-0.5)]$$

$$X_3(\Omega) = 2\frac{X(\Omega/2)}{2} = X(\Omega/2)$$

(d) Modulation: $x_4(t) = \cos(0.5\pi t)x(t)$ so

$$X_4(\Omega) = 0.5[X(\Omega + 0.5\pi) + X(\Omega - 0.5\pi)]$$

(e) $x_5(t) = X(t)$ so that by duality

$$X_5(\Omega) = 2\pi x(-\Omega) = 2\pi[u(-\Omega+1) - u(-\Omega-1)] = 2\pi x(\Omega)$$

5.6 (a) $x(t) = \cos(t)[u(t) - u(t-1)] = \cos(t)p(t)$, so

$$X(\Omega) = 0.5[P(\Omega + 1) + P(\Omega - 1)]$$

where

$$P(\Omega) = \frac{e^{-s/2}(e^{s/2} - e^{-s/2})}{s}|_{s=j\Omega} = 2e^{-j\Omega/2}\frac{\sin(\Omega/2)}{\Omega}$$

(b) $y(t) = x(2t) = \cos(2t)p(2t) = \cos(2t)[u(t) - u(t-0.5)]$, so

$$Y(\Omega) = 0.5[P_1(\Omega + 2) + P_1(\Omega - 2)]$$

where

$$P_1(\Omega) = \mathcal{F}[u(t) - u(t-0.5)] = 2e^{-j\Omega/4}\frac{\sin(\Omega/4)}{\Omega}$$

$z(t) = x(t/2) = \cos(t/2)p(t/2) = \cos(t/2)[u(t) - u(t-2)] = \cos(t/2)p_2(t)$ so

$$Z(\Omega) = 0.5[P_2(\Omega + 0.5) + P_2(\Omega - 0.5)]$$

$$P_2(\Omega) = \mathcal{F}[u(t) - u(t-2)] = 2e^{-j\Omega}\frac{\sin(\Omega)}{\Omega}$$

Using

$$P_1(\Omega) = 0.5P(\Omega/2)$$
$$P_2(\Omega) = 2P(2\Omega)$$

we have

$$\begin{aligned} X(\Omega) &= 0.5[P(\Omega+1) + P(\Omega-1)] \\ Y(\Omega) &= 0.5[0.5P((\Omega/2)+1) + 0.5P((\Omega/2)-1)] = 0.5X(\Omega/2) \\ Z(\Omega) &= 0.5[2P(2\Omega+1) + 2P(2\Omega-1)] = 2X(2\Omega) \end{aligned}$$

5.7 $\mathcal{F}[\delta(t-\tau)] = \mathcal{L}[\delta(t-\tau)]_{s=j\Omega} = e^{-j\Omega\tau}$ so

(a) By linearity and time–shift

$$\mathcal{F}[\delta(t-1)+\delta(t+1)] = 2\cos(\Omega)$$

(b) By duality

$$\begin{aligned} 0.5[\delta(t-\tau)+\delta(t+\tau)] &\quad\leftrightarrow\quad \cos(\Omega\tau) \\ \cos(\Omega_0 t) &\quad\leftrightarrow\quad \pi[\delta(\Omega+\Omega_0)+\delta(\Omega-\Omega_0)] \end{aligned}$$

by letting $\tau = \Omega_0$ in the second equation.

(c) Considering

$$\mathcal{F}[\delta(t-1)-\delta(t+1)] = 2j\sin(\Omega),$$

by duality

$$\begin{aligned} -0.5j[\delta(t-\tau)+\delta(t+\tau)] &\quad\leftrightarrow\quad \sin(\Omega\tau) \\ \sin(\Omega_0 t) &\quad\leftrightarrow\quad -j\pi[\delta(\Omega+\Omega_0)+\delta(\Omega+\Omega_0)] \end{aligned}$$

by letting $\tau = \Omega_0$ in the second equation.

5.14 (a) Let $X(\Omega) = A[u(\Omega + \Omega_0) - u(\Omega - \Omega_0)]$ its inverse Fourier transform is

$$x(t) = \frac{1}{2\pi}\int_{-\Omega_0}^{\Omega_0} Ae^{j\Omega t}d\Omega = \frac{A\sin(\Omega_0 t)}{\pi t}$$

so $A = 1$, $\Omega_0 = 0.5$ and $X(\Omega) = u(\Omega + 0.5) - u(\Omega - 0.5)$.

(b) $Y(\Omega) = H(\Omega)X(\Omega) = X(\Omega)$ so that $y(t) = (x * x)(t) = x(t)$, or convolution of a sinc function with itself is a sinc.

5.17 (a) Impulse response

$$\begin{aligned} h(t) &= \frac{1}{2\pi}\int_{-2}^{2} 1e^{j\angle H(j\Omega)}e^{j\Omega t}d\Omega = \frac{1}{2\pi}\int_{0}^{2} e^{j(\Omega t-\pi/2)}d\Omega + \frac{1}{2\pi}\int_{-2}^{0} e^{j(\Omega t+\pi/2)}d\Omega \\ &= \frac{-j}{2\pi jt}(e^{j2t}-1) - \frac{j}{2\pi jt}(e^{-j2t}-1) = \frac{1-\cos(2t)}{\pi t} \end{aligned}$$

(b) The frequency components of $x(t)$ with harmonic frequencies bigger than 2 are filtered out so

$$y_{ss}(t) = 2|H(j1.5)|\cos(1.5t + \angle H(j1.5)) = 2\cos(1.5t - \pi/2) = 2\sin(1.5t)$$

5.18 (a) Plot of $X(\Omega)$ as function of Ω:

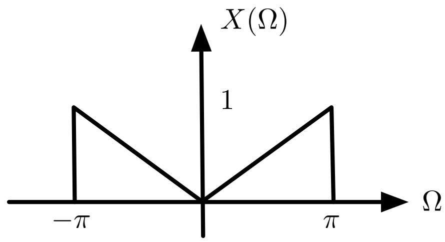

Figure 5.3: Problem 18

(b)

$$x(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\Omega|}{\pi}d\Omega = \frac{2}{2\pi}\int_{0}^{\pi}\frac{\Omega}{\pi}d\Omega = \frac{1}{2}$$

5.19 (a) Poles are roots of $D(s) = s^2 + 2s + 2 = (s+1)^2 + 1 = 0$ or

$$s_{1,2} = -1 \pm j1$$

the zero is $s = 0$. It is a band-pass filter with center frequency around 1. Its magnitude response is using vectors from the zero and the poles to the point in the $j\Omega$–axis where are finding the frequency response:

Ω	$\lvert H(j\Omega)\rvert$
0	0 (zero at zero)
1	$\sqrt{5}(1)/[(1)(\sqrt{4+1})] = 1$
∞	0 (vectors of two poles and zero have infinite lengths)

(b) Impulse response

$$\begin{aligned} H(s) &= \frac{\sqrt{5}(s+1)}{(s+1)^2+1} - \frac{\sqrt{5}}{(s+1)^2+1} \\ h(t) &= \sqrt{5}e^{-t}\left(\cos(t) - \sin(t)\right)u(t) = \sqrt{5}e^{-t}\sqrt{2}\cos(t+\pi/4)u(t) \\ &= \sqrt{10}e^{-t}\cos(t+\pi/4)u(t) \end{aligned}$$

(c) The steady state response corresponding to $x(t) = B + \cos(\Omega t)$ is

$$\begin{aligned} y(t) &= B|H(j0)| + |H(j\Omega_0)|\cos(\Omega_0 + \angle H(j\Omega_0)) \\ &= |H(j\Omega_0)|\cos(\Omega_0 + \angle H(j\Omega_0)) \end{aligned}$$

for Ω_0 to be determined by looking at frequencies for which

$$|H(j\Omega_0)| = \frac{\sqrt{5}\Omega_0}{\sqrt{(2-\Omega_0^2)^2 + 4\Omega_0^2}} = 1 \qquad \text{or}$$
$$5\Omega_0^2 = 4 - 4\Omega_0^2 + \Omega_0^4 + 4\Omega_0^2 \quad \Rightarrow \quad \Omega_0^4 - 5\Omega_0^2 + 4 = (\Omega_0^2 - 4)((\Omega_0^2 - 1) = 0$$

giving values of

$$\Omega_0 = \pm 2, \ \pm 1$$

so we have that when $\Omega = 1$ or 2 the dc bias is filtered out and the cosine has a magnitude of 1. The corresponding phases are using the pole and zero vectors

$$\begin{aligned} \Omega_0 = 1 \quad &\Rightarrow \quad \angle H(j\Omega_0) = \pi/2 - 0 - \tan^{-1}(2) \\ \Omega_0 = 2 \quad &\Rightarrow \quad \angle H(j\Omega_0) = \pi/2 - \tan^{-1}(1) - \tan^{-1}(3) \end{aligned}$$

5.22 (a) According to the eigenvalue property for $x(t) = e^{j\Omega t}$, $-\infty < \Omega < \infty$, the output in the steady-state would be $y(t) = e^{j\Omega t}H(j\Omega)$ so that the differential equation gives

$$j\Omega e^{j\Omega t}H(j\Omega) = -e^{j\Omega t}H(j\Omega) + e^{j\Omega t}$$

$$\text{giving } H(j\Omega) = \frac{1}{1+j\Omega}$$

$$|H(j\Omega)| = \frac{1}{\sqrt{1+\Omega^2}}, \quad \angle H(j\Omega) = -\tan^{-1}(\Omega)$$

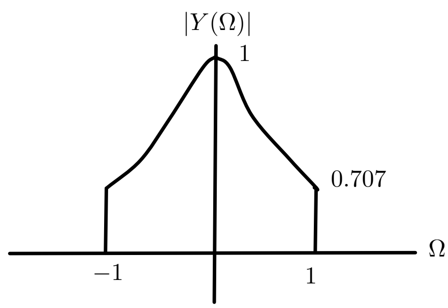

Figure 5.4: Problem 22

(b) The magnitude response indicates the filter is a low-pass filter, in particular

Ω	$\lvert H(j\Omega)\rvert$	$\angle H(j\Omega)$
0	1	0
1	$\frac{1}{\sqrt{2}}$	$-\pi/4$
∞	0	$-\pi/2$

(c) The Fourier transform of $x(t)$ is $X(\Omega) = u(\Omega+1) - u(\Omega-1)$ so that the Fourier transform of the output is

$$Y(\Omega) = X(\Omega)H(j\Omega)$$

with magnitude response as in Fig. 5.4