4.2 (a) Replacing $x(\tau) = e^{j\Omega_0 \tau}$ in the integral for the averager we get

$$\begin{aligned}\frac{1}{T}\int_{t-T}^{t} e^{j\Omega_0\tau} d\tau &= \frac{1}{T}\frac{e^{j\Omega_0 t} - e^{j\Omega_0(t-T)}}{j\Omega_0} \\ &= e^{j\Omega_0 t}\left[\frac{1-e^{-j\Omega_0 T}}{j\Omega_0 T}\right]\end{aligned}$$

where according to the eigenfunction property $H(j\Omega_0)$, the frequency response of the system for Ω_0, is the term in the square brackets.

(b) By a change of variable, $\sigma = t - \tau$, the equation for the averager is

$$y(t) = \frac{1}{T}\int_{0}^{T} x(t-\sigma) d\sigma$$

which is a convolution integral with impulse response $h(t) = (1/T)[u(t) - u(t-T)]$. The transfer function of the system is $H(s) = (1/sT)[1 - e^{-sT}]$ and if we let $s = j\Omega_0$ we get the above response.

4.4 (a) $x(t)$ is a train of pulses with $T_0 = 2\pi$ and $\Omega_0 = 1$. The Fourier series coefficients are

$$\begin{aligned}
X_0 &= 0 \text{ by symmetry in a period} \\
X_k &= \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-1)e^{-jkt}dt + \int_{0}^{\pi}(1)e^{-jkt}dt\right] = \frac{1}{2\pi}\int_{0}^{\pi}(e^{-jkt} - e^{jkt})dt \\
&= \frac{1}{2\pi}\left[\frac{-(e^{-j\pi k} + e^{j\pi k} - 2)}{jk}\right] = \frac{1 - \cos(\pi k)}{j\pi k}
\end{aligned}$$

since $\cos(\pi k) = (-1)^k$ we get that

$$X_k = \begin{cases} 0 & k \text{ even} \\ -2j/(\pi k) & k \text{ odd} \end{cases}$$

(b) Laplace transform of a period

$$X_1(s) = \frac{1}{s}(1 - 2e^{-\pi s} + e^{-2\pi s}) = \frac{2e^{-\pi s}}{s}(\cosh(\pi s) - 1)$$

so that the Fourier series coefficients are

$$X_k = \frac{1}{2\pi}X_1(s)|_{s=jk} = j\,\frac{(-1)^{k+1}(\cos(\pi k) - 1)}{\pi k}$$

or

$$X_k = \begin{cases} 0 & k \;\; \text{even} \\ -2j/(\pi k) & k \;\; \text{odd} \end{cases}$$

4.6 (a) A period is $x_1(t) = t[u(t) - u(t-1)]$, its fundamental frequency $\Omega_0 = 2\pi$, and its fundamental period $T_0 = 1$. See Fig. 4.2.

(b) Fourier series coefficients using the integral definition:

$$\begin{aligned} X_k &= \frac{1}{T_0}\int_0^1 te^{-j2\pi kt}dt = \frac{e^{-j2\pi kt}}{(-j2\pi k)^2}(-j2\pi kt - 1)|_{t=0}^1 \\ &= \frac{j2\pi k + 1}{4\pi^2 k^2} - \frac{1}{4\pi^2 k^2} = \frac{j}{2\pi k} \qquad k \neq 0 \end{aligned}$$

If $k = 0$, the dc value X_0 is

$$X_0 = \frac{1}{T_0}\int_0^1 tdt = \frac{t^2}{2}|_{t=0}^1 = 0.5$$

(c) Since $x_1(t) = tu(t) - tu(t-1)$ with Laplace transform

$$\begin{aligned} &X_1(s) = \frac{1}{s^2} - \mathcal{L}[tu(t-1)] \ \text{ where} \\ &\mathcal{L}[tu(t-1)] = \mathcal{L}[(t-1)u(t-1) + u(t-1)] = \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s} \end{aligned}$$

The FS coefficients are

$$\begin{aligned} X_k &= \frac{1}{T_0}\mathcal{L}[x_1(t)]|_{s=j2\pi k} = \frac{1-e^{-s}}{s^2} - \frac{e^{-s}}{s}|_{s=j2\pi k} \\ &= \frac{1-e^{-j2\pi k}}{(j2\pi k)^2} - \frac{e^{-j2\pi k}}{j2\pi k} = j\frac{1}{2\pi k} \qquad k \neq 0 \end{aligned}$$

By inspection, the mean is $X_0 = 0.5$ (it cannot be calculated using the Laplace transform method). Notice that the zero-mean signal is odd so the X_k are purely imaginary.

(d) The derivative of $x(t)$ is (see bottom plot in Fig. 4.2):

$$y(t) = \frac{dx(t)}{dt} = 1 - \sum_{k=-\infty}^{\infty}\delta(t-k)$$

A period of it is given by $y_1(t) = u(t+0.5) - u(t-0.5) - \delta(t)$ and by the Laplace transform we have the Fourier series coefficients of $y(t)$ of fundamental period $T_0 = 1$ are

$$\begin{aligned} &Y_k = \left(\frac{e^{0.5s} - e^{-0.5s}}{s} - 1\right)|_{s=j2\pi k} = \frac{\sin(k\pi)}{k\pi} - 1 = -1 \ \text{ and so} \\ &\text{using the derivative property } \ X_k = \frac{-1}{j2\pi k} = \frac{j}{2\pi k} \ \ k \neq 0 \end{aligned}$$

which coincide with the ones obtained before. The X_0 cannot be calculated from above.

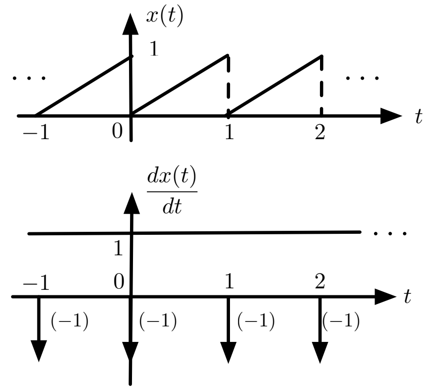

Figure 4.2: Problem 6

4.7 (a) i. $T_0 = 2$ since $\Omega_0 = \pi$.

ii. dc value $X_0 = 3/4$

iii. $x(t)$ is even, since the X_k are real.

iv. For the third harmonic

$$\frac{3}{4+9\pi^2}(e^{j3\pi t} + e^{-j3\pi t}) = \frac{6}{4+9\pi^2}\cos(3\pi t)$$

then $A = 6/(4+9\pi^2)$.

(b) Since $x(1) = 1$ then letting $t = 1$ we have

$$\pi = 4\sum_{k=1}^{\infty} \frac{1}{2k-1}\sin(2k-1)$$

It is also possible to find a similar expression for other values of t which are not in the discontinuities.

4.9 (a) If the periodic signal

$$x(t) = \sum_k X_k e^{j\Omega_0 kt}$$

then

$$y(t) = 2x(t) - 3 = (2X_0 - 3) + \sum_{k \neq 0} 2X_k e^{j\Omega_0 kt}$$

is also periodic of period T_0 with Fourier coefficients

$$Y_k = \begin{cases} 2X_0 - 3 & k = 0 \\ 2X_k & k \neq 0 \end{cases}$$

The signal

$$\begin{aligned} z(t) &= x(t-2) + x(t) \\ &= \sum_k X_k e^{j\Omega_0 k(t-2)} + \sum_k X_k e^{j\Omega_0 kt} \\ &= \sum_k [X_k(1 + e^{-2j\Omega_0 k})] e^{j\Omega_0 kt} \end{aligned}$$

is periodic of period T_0 and with Fourier series coefficients $Z_k = X_k(1 + e^{-2j\Omega_0 k})$.
The signal

$$w(t) = x(2t) = \sum_k X_k e^{j\Omega_0 k2t} = \sum_{m \text{ even}} X_{m/2} e^{j\Omega_0 mt}$$

is periodic of period $T_0/2$, with Fourier series coefficients

$$W_k = \begin{cases} X_{k/2} & k \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

4.10 (a) The sinusoidal components have periods $T_1 = 1$ and $T_2 = 1/2$ so that $T_2/T_1 = 1/2$. $x(t)$ is periodic of fundamental period $T_1 = 2T_2 = 1$ and $\Omega_0 = 2\pi$.

(b) Expressing

$$x(t) = 0.5 + 2(e^{j\Omega_0 t} + e^{-j\Omega_0 t}) - 4(e^{j2\Omega_0 t} + e^{-j2\Omega_0 t})$$

we get $X_0 = 0.5$, $X_1 = X_{-1}^* = 2$ and $X_2 = X_{-2}^* = -4$ (See Fig. 4.3). Plotting $|X_k|^2$ indicates the power at each of the harmonics and it can be seen that the highest power is at $2\Omega_0 = 4\pi$ rad/sec.

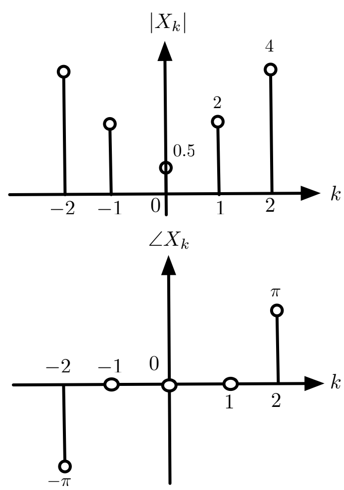

Figure 4.3: Problem 10

(c) Comparing

$$y(t) = 2 - 2\sin(2\pi t) = 2 + 2\cos(2\pi t + \pi/2)$$

with the output obtained from applying the eigenfunction property of LTI:

$$\begin{aligned} y(t) &= 0.5|H(j0)| + 4|H(j2\pi)|\cos(2\pi t + \angle H(j2\pi)) \\ &\quad -8|H(j4\pi)|\cos(4\pi t + \angle H(j4\pi)) \end{aligned}$$

gives $H(j0) = 4$, $H(j2\pi) = 0.5e^{j\pi/2}$ and $H(j4\pi) = 0$.

which gives

$$\sqrt{A^2+B^2} = 1$$
$$\tan^{-1}(B/A) = -2$$

which are satisfied by $A=\cos(2)$ and $B=-\sin(2)$, indeed $A^2+B^2=(\cos(2))^2+(\sin(2))^2=1$ and $\tan^{-1}(B/A)=\tan^{-1}[-\tan(2)]=-2$. So that the Laplace transform

$$\begin{aligned}\mathcal{L}[\cos(t)u(t-2)] &= \mathcal{L}[(A\cos(t-2)+B\sin(t-2))u(t-2)]\\ &= A\mathcal{L}[\cos(t-2)u(t-2)]+B\mathcal{L}[\sin(t-2)u(t-2)]\\ &= \frac{e^{-2s}(\cos(2)s-\sin(2))}{s^2+1}\end{aligned}$$

which coincides with the result using the Laplace transform obtained before.

4.12 (a) The steady state is

$$y(t) = 4|H(j2\pi)|\cos(2\pi t + \angle H(j2\pi)) + 8|H(j3\pi)|\cos(3\pi t - \pi/2 + \angle H(j3\pi))$$

so $H(j2\pi) = 0.5e^{j\pi} = -0.5$, $H(j3\pi) = 0$. Nothing else can be learned about the filter from the input/output.

(b)

$$\begin{aligned} y_{ss}(t) &= \sum_{k=1}^{\infty} \frac{2}{k^2}|H(j3k/2)|\cos(3kt/2 + \angle H(j3k/2)) \\ &= 2|H(j3/2)|\cos(3t/2 + \angle H(j3/2)) \\ &= 2\cos(3t/2 - \pi/2) \end{aligned}$$

since for frequencies bigger than 2 the magnitude response is zero.

4.13 (a) $g_1(t) = dx_1(t)/dt = -0.5[u(t) - u(t-1)] + 0.5\delta(t-1)$ so that

$$g(t) = \sum_k (-0.5[u(t-k) - u(t-k-1)]) + \sum_k 0.5\delta(t-k-1) = -0.5 + \sum_k 0.5\delta(t-k-1)$$

periodic of fundamental period$T_0 = 1$.

(b) Fourier series of $g(t)$ and $x(t)$

$$\begin{aligned}
g(t) &= \frac{dx(t)}{dt} = \sum_{k=-\infty,\neq 0}^{\infty} X_k jk\Omega_0 e^{jk\Omega_0 t} \\
G_k &= jk\Omega_0 X_k \quad \text{(derivative property) } \Omega_0 = 2\pi \\
&= G_1(s)|_{s=jk2\pi} = -0.5\frac{1-e^{-s}}{s} + 0.5e^{-s}|_{s=jk2\pi} = 0.5 \quad \text{(definition)} \\
X_k &= \frac{0.5}{jk2\pi} \quad k \neq 0
\end{aligned}$$

The dc term cannot be obtained from $g(t)$, it is calculated by

$$X_0 = \int_0^1 (-0.5t)dt = -0.25$$

(c) Fourier series of

$$y(t) = 0.5 + x(t) = 0.25 + \sum_{k=-\infty,\neq 0}^{\infty} X_k e^{j2\pi kt}$$

the only difference is the dc value.

4.17 (a) The signal in $[0,\ 1]$ is

$$x_1(t) = u(t) - r(t) + r(t-1)$$

so that ($T_0 = 1$, $\Omega_0 = 2\pi$)

$$\begin{aligned} X_k &= \left[\frac{1}{s} - \frac{1}{s^2}(1-e^{-s})\right]_{s=j2\pi k} = \left[\frac{1}{s} - \frac{e^{-s/2}}{s^2}(e^{s/2} - e^{-s/2})\right]_{s=j2\pi k} \\ &= \frac{-j}{2\pi k} + \frac{e^{-j\pi}}{4\pi^2 k^2} 2j\sin(\pi k) = \frac{-j}{2\pi k} \qquad k \neq 0 \end{aligned}$$

and by inspection $X_0 = 0.5$. Thus the Fourier series for $x(t)$ is

$$x(t) = 0.5 + \sum_{k=-\infty,\neq 0}^{\infty} \frac{-j}{2\pi k} e^{jk2\pi t}$$

(b) The derivative of $x(t)$ (from the figure) is

$$g(t) = \frac{dx(t)}{dt} = -1 + \sum_{k=-\infty}^{\infty} \delta(t-k)$$

with a period between -0.5 and 0.5 of

$$g_1(t) = -u(t+0.5) + u(t-0.5) + \delta(t)$$

and same fundamental frequency Ω_0 as $x(t)$ thus

$$G_k = \left[\frac{-1}{s}(e^{s/2} - e^{-s/2}) + 1\right]_{s=j2\pi k} = \frac{j}{2\pi k} 2j\sin(\pi k) + 1 = 0 + 1 = 1$$

so that

$$g(t) = \sum_{k=-\infty}^{\infty} e^{jk2\pi t}$$

equal to the derivative of the Fourier series of $x(t)$. Using the derivative Fourier series property

$$G_k = j\Omega_0 k X_k \;\; \Rightarrow \;\; X_k = \frac{1}{j2\pi k} = \frac{-j}{2\pi k}$$

Using these and $X_0 = 0.5$ we can then get the Fourier series for $x(t)$.

4.18 (a) According to the eigenfunction property of LTI the steady state response corresponding to the given $x(t) = 1 + \cos(t + \pi/4) = \cos(0t) + \cos(t + \pi/4)$ is

$$y_{ss}(t) = |H(j0)| \cos(0t + \angle H(j0)) + |H(j)| \cos(t + \pi/4 + \angle H(j))$$

Since

$$H(j0) = H(s) \mid_{s=j0} = \frac{1}{2} e^{j0}$$
$$H(j) = H(s) \mid_{s=j} = \frac{1+j}{1+3j} = \frac{4-2j}{10} = 0.447\angle{-26.6^o}$$

$$y_{ss}(t) = 0.5 + 0.447 \cos(t + 18.4^o)$$

(b) i. The input $x(t) = 4u(t) = 4\cos(0t)u(t)$ in the steady state is $4\cos(0t)$, i.e., a cosine of frequency zero, so that its response is

$$y_{ss}(t) = 4|H(j0)| \cos(0t + \angle H(j0)) = 4 \times 0.5 = 2$$

ii. If $x(t) = 4u(t)$, then in the Laplace transform

$$Y(s) = \frac{4}{s((s+1.5)^2 + (2 - 1.5^2))} = \frac{A}{s} + \cdots$$

where the $\cdots$ stands for terms that have poles in the left-hand s-plane that correspond to the transient so

$$y_{ss}(t) = A = Y(s)s \mid_{s=0} = 2$$

4.21 (a) The derivative of the period between 0 and 2 of the triangular signal $x(t)$ is

$$y_1(t) = \frac{dx_1(t)}{dt} = u(t) - 2u(t-1) + u(t-2)$$

and the signal

$$y(t) = \sum_k y_1(t-2k)$$

is a train of square pulses of period $T_0 = 2$ and average zero. The signal $z(t) = y(t) + 1$ is also periodic of the same period as $y(t)$ but average 1. The Fourier series coefficients of $y(t)$ are

$$\begin{aligned} Y_k &= \frac{1}{2s}e^{-s}(e^s - 2 + e^{-s})|_{s=j\pi k} = \frac{1}{j\pi k}e^{-j\pi k}(\cos(\pi k) - 1) \\ &= \frac{1}{j\pi k}\left[(-1)^{2k} - (-1)^k\right] = \frac{2}{j\pi k} \qquad k \neq 0, \text{ odd} \end{aligned}$$

and zero for k even. The Fourier coefficients of $z(t)$ are 1 as d.c. value and $Z_k = Y_k$ for k odd and zero for k even and $\neq 0$.
(b) The Fourier series of $y(t)$ is

$$\begin{aligned} y(t) &= \sum_{k=-\infty, \text{ odd}}^{\infty} \frac{2}{j\pi k}e^{j\pi kt} \\ &= 4\sum_{k>0, \text{ odd}} \frac{\sin(\pi kt)}{\pi k} \end{aligned}$$

and that of the signal $z(t)$ is

$$z(t) = 1 + 4\sum_{k>0, \text{ odd}} \frac{\sin(\pi kt)}{\pi k}$$

The signal $y(t)$ is an odd function of t and as such it can be represented by sines, and this is why it has purely imaginary coefficients for its exponential Fourier series. The signal $z(t)$ is neither even nor odd and as such it is made up of an even component, the constant 1, and an odd component, $y(t)$.
(c) If we reverse the process in (a) by integrating $y_1(t)$ we get $x_1(t) = r(t) - 2r(t-1) + r(t-2)$ which would give us the periodic signal $x(t)$ by shifting and adding. The Fourier series coefficients of $x(t)$ are

$$X_k = \frac{Y_k}{j\pi k} = \frac{2}{-(\pi k)^2} \qquad k \neq 0, \text{ odd, and } \quad X_0 = 0.5$$

and zero for k even. Notice that the integral of $y(t)$ would be zero for each period and would give the $x_1(t)$ shifted in time in each period.
(d) The signal $x(t)$ is even and as such it can be represented by a cosine Fourier series so the coefficients are real. We have that

$$\begin{aligned} x(t) &= 0.5 + \sum_{k=-\infty,\ k\neq 0, \text{ odd}}^{\infty} \frac{-2}{(\pi k)^2}e^{j\pi kt} \\ &= 0.5 + \sum_{k=1, \text{ odd}}^{\infty} \frac{4\cos(\pi kt - \pi)}{(\pi k)^2} \end{aligned}$$