Chapter 3

The Laplace Transform

3.1 Basic Problems

3.1 (a) Finite–support signals.

i. $\mathcal{L}[x(t)] = \int_{-\infty}^{\infty} \delta(t-1)e^{-st}dt = e^{-s}$, ROC the whole s-plane

ii. $\mathcal{L}[y(t)] = e^{s} - e^{-s} = 2\sinh(s)$, ROC the whole s-plane

iii. We have

$$\mathcal{L}[z(t)] = \mathcal{L}[u(t+1)] - \mathcal{L}[u(t-1)] = \frac{e^{s} - e^{-s}}{s} = 2\frac{\sinh(s)}{s}$$

ROC whole s-plane, pole-zero cancellation at $s = 0$.

iv. Notice that because $\cos(2\pi(t \pm 1)) = \cos(2\pi t \pm 2\pi) = \cos(2\pi t)$ then

$$w(t) = \cos(2\pi t)[u(t+1) - u(t-1)] = \cos(2\pi(t+1))u(t+1) - \cos(2\pi(t-1))u(t-1)$$

so that

$$W(s) = \frac{e^{s}s}{s^2+4\pi^2} - \frac{e^{-s}s}{s^2+4\pi^2} = \frac{(e^{s}-e^{-s})s}{s^2+4\pi^2} = \frac{2s\sinh(s)}{s^2+4\pi^2}$$

The poles $p_{1,2} = \pm j2\pi$ are cancelled by zeros $z_{1,2} = \pm j2\pi$ (indeed, $e^{s} - e^{-s}|_{s=\pm j2\pi} = 1 - 1 = 0$) so the ROC is the whole s-plane.

(b) Causal signals.

i. $X_1(s) = \mathcal{L}[x_1(t)] = 1/(s+1)$, ROC $\sigma > -1$

ii. Notice the exponential is not delayed, so the shifting property cannot be applied. Instead,

$$Y_1(s) = \mathcal{L}[y_1(t)] = \mathcal{L}[e^{-1}e^{-(t-1)}u(t-1)] = \frac{e^{-(s+1)}}{s+1}, \quad \text{ROC } \sigma > -1$$

iii. $\mathcal{L}[z_1(t)] = \mathcal{L}[e^{-(t-1)}u(t-1)] = e^{-s}/(s+1)$, ROC $\sigma > -1$

iv. Since $w_1(t) = x_1(t) - y_1(t)$, where $x_1(t)$ and $y_1(t)$ are given before then

$$\mathcal{L}[w_1(t)] = X_1(s) - Y_1(s) = \frac{1 - e^{-(s+1)}}{s+1},$$

ROC whole s-plane because pole $s = -1$ is cancelled by zero, i.e., $1 - e^{-(s+1)}|_{s=-1} = 0$.

3.2 (a) Anti–causal signals.

i. Using the integral definition of Laplace transform:

$$X(s) = \int_{-\infty}^{0} e^{t} e^{-st} dt = -\int_{\infty}^{0} e^{-\tau} e^{s\tau} d\tau = \int_{0}^{\infty} e^{(s-1)\tau} d\tau = \frac{-1}{s-1} = \frac{1}{1-s}$$

by letting $\tau = -t$. The ROC is $\sigma < 1$ to make upper limit of the integral go to zero.

ii. Let $v(t) = y(-t) = e^{-t}u(t-1)$ with $V(s) = e^{-1}e^{-s}/(s+1)$, ROC: $\sigma > -1$ then

$$Y(s) = V(-s) = \frac{e^{(s-1)}}{-s+1}, \qquad \text{ROC: } \sigma < 1.$$

A direct way is

$$\begin{aligned} Y(s) &= \int_{-\infty}^{-1} e^{t} e^{-st} dt \quad \text{letting } t' = -t \\ &= \int_{1}^{\infty} e^{-t'(1-s)} dt' = \frac{e^{s-1}}{1-s} \qquad \text{ROC } 1 - \sigma > 0 \text{ or } \sigma < 1 \end{aligned}$$

iii. We have

$$\begin{aligned} Z(s) &= \int_{-\infty}^{\infty} e^{t+1} u(-(t+1)) e^{-st} dt \;\text{ letting } \rho = -(t+1) \\ &= \int_{-\infty}^{\infty} e^{-\rho} u(\rho) e^{s(\rho+1)} d\rho = \int_{0}^{\infty} e^{s} e^{\rho(s-1)} d\rho = e^{s} \frac{e^{\rho(s-1)}}{s-1}\Big|_{0}^{\infty} = \frac{e^{s}}{-s+1} \end{aligned}$$

ROC: $\sigma < 1$ to make the upper limit of the integral go to zero.

iv. $w(t) = e^{t}u(-t) - e^{t}u(-t-1) = x(t) - y(t)$,

$$W(s) = -\frac{1}{s-1} + \frac{e^{(s-1)}}{s-1} = \frac{e^{(s-1)} - 1}{s-1}$$

ROC: whole s-plane, since pole $s = 1$ is cancelled by zero at $s = 1$(indeed, $e^{(s-1)} - 1|_{s=1} = 0$). Notice that due to the pole–zero cancellation the region of convergence is not the intersection of the ROCs of $X(s)$ and $Y(s)$

(b) Non–causal signals.

i. $X_1(s) = (e^{s} - e^{-s})/s = 2\sinh(s)/s$, ROC whole s-plane because pole at $s = 0$ is cancelled by zero at $s = 0$ (since $\sinh(0) = 0$).

ii. $y_1(t) = e^{-t}u(t+1) = e[e^{-(t+1)}u(t+1)]$ so

$$Y_1(s) = \frac{e^{(s+1)}}{s+1}, \;\text{ ROC: } \sigma > -1.$$

iii. We have

$$\begin{aligned} Z_1(s) &= \mathcal{L}[(e^{(t+1)}u(t+1))e^{-1}] - \mathcal{L}[(e^{(t-1)}u(t-1))e] = \frac{e^{s-1} - e^{-(s-1)}}{s-1} \\ &= \frac{2\sinh(s-1)}{s-1} \;\text{ ROC: whole s-plane} \end{aligned}$$

because there is pole/zero cancellation at $s = 1$.

3.4 (a) i. For $\alpha > 0$,

$$X(s) = \frac{1}{s+\alpha} - \frac{1}{-s+\alpha} = \frac{-2s}{\alpha^2 - s^2} = \frac{2s}{s^2-\alpha^2} \quad \text{ROC: } -\alpha < \sigma < \alpha$$

where the ROC is the intersection of ROC of $1/(s+\alpha)$, or $\sigma > -\alpha$, and the ROC of $1/(-s+\alpha)$, or $\sigma < \alpha$.

ii. As $\alpha \to 0$, $x(t) = u(t) - u(-t)$, the sign signal, which has no Laplace transform because ROC does not exist.

(b) For the sampled signal

$$X_1(s) = \sum_{n=0}^{N-1} e^{-2n}\mathcal{L}[\delta(t-n)] = \sum_{n=0}^{N-1} e^{-(s+2)n} = \frac{1-e^{-(s+2)N}}{1-e^{-(s+2)}}$$

The poles of $X_1(s)$ are values of s that make $e^{-(s+2)} = 1 = e^{j2\pi k}$, or $s_k = -2 - j2\pi k$ for $k = 0, \pm 1, \pm 2 \cdots$. Similarly, the zeros of $X_1(s)$ are $s_m = -2 - j2\pi(m/N)$ for $m = 0, \pm 1, \pm 2 \cdots$. Because the poles are cancelled when $m = 0, \pm N, \pm 2N, \cdots$ (i.e., multiples of N), the ROC of $X_1(s)$ is the whole s-plane.

(c) We have

$$S(s) = \sum_{n=0}^{\infty} \mathcal{L}[u(t-n)] = \sum_{n=0}^{\infty} \frac{e^{-ns}}{s} = \frac{1}{s(1-e^{-s})}$$

The poles are $s = 0$ and values of s that make $1 - e^{-s} = 0$ or $s_k = -j2\pi k$ for $k = 0, \pm 1, \cdots$. Thus, all the poles are on the $j\Omega$-axis and the signal is causal, as such the ROC is $\sigma > 0$.

(d) Since $\sin(2\pi t) = \sin(2\pi t - 2\pi) = \sin(2\pi(t-1))$ we have

$$v(t) = [\cos(2(t-1)) + \sin(2\pi(t-1))]u(t-1)$$

then

$$V(s) = \frac{e^{-s}s}{s^2+4} + \frac{2\pi e^{-s}}{s^2+4\pi^2}, \qquad \text{ROC: } \sigma > 0$$

(e) The Laplace transform of $y(t)$ is

$$Y(s) = \int_{-\infty}^{\infty} t^2 e^{-2t} u(t) e^{-st} dt = \int_{-\infty}^{\infty} (t^2 u(t)) e^{-(s+2)t} dt = \mathcal{L}[t^2 u(t)]_{s \to s+2} = \frac{2}{(s+2)^3}$$

with ROC $\sigma + 2 > 0$ or $\sigma > -2$.

3.5 (a)

$$\mathcal{L}[x(t)e^{-at}u(t)] = \int_{-\infty}^{\infty} [x(t)u(t)]e^{-(s+a)t}dt = X(s+a)$$

i.e., multiplying by e^{-at} shifts the complex frequency s to $s+a$. This property is called frequency shift. Thus,

$$Y(s) = \mathcal{L}[(\cos(t)u(t))e^{-2t}] = \mathcal{L}[\cos(t)u(t)]_{s \to s+2} = \frac{s+2}{(s+2)^2+1}$$

(b) The zero is $s = -2$ and the poles $s_{1,2} = -2 \pm j$. Because the poles are in the left-hand s-plane $x_1(t) \to 0$ as $t \to \infty$. Indeed, we have that $X_1(s)$ is the Laplace transform of $x_1(t) = \cos(t)e^{-2t}u(t)$ which tends to zero as $t \to \infty$.

(c) i. $z(t) = -e^{-t}u(t) + \delta(t)$, so

$$Z(s) = \frac{-1}{s+1} + 1 = \frac{s}{s+1}$$

ii. If $f(t) = e^{-t}u(t)$, then $z(t) = df(t)/dt$ so

$$Z(s) = s\mathcal{L}[f(t)] - f(0-) = \frac{s}{s+1} - 0 = \frac{s}{s+1}.$$

3.7 (a) The Laplace transform of $u(-t)$ is $-1/s$, indeed

$$\mathcal{L}[u(-t)] = \int_{-\infty}^{0} e^{-st} dt = \int_{0}^{\infty} e^{s\tau} d\tau = \frac{e^{s\tau}}{s}|_{\tau=0}^{\infty} = -\frac{1}{s}$$

provided that $\sigma < 0$ so that the limit when $\tau = \infty$ is zero. Thus

$$\mathcal{L}[u(-t)] = \frac{-1}{s} \qquad \text{ROC}: \quad \sigma < 0$$

Thus $x(t) = u(t) + u(-t) = 1$ for all t would have as Laplace transform

$$X(s) = \frac{1}{s} - \frac{1}{s} = 0$$

and as ROC the intersection of $\sigma > 0$ and $\sigma < 0$ which is null. So we cannot find the Laplace of $x(t) = 1$, $-\infty < t < \infty$.

(b) Let $y(t) = y_c(t) + y_{ac}(t)$ where the causal component $y_c(t) = y(t)u(t) = e^{-t}u(t)$ and the anticausal component $y_{ac}(t) = y(t)u(-t) = e^{t}[u(t+1) - u(t)]$. The Laplace transform of $y_{ac}(t)$ is

$$Y_{ac}(s) = \int_{-1}^{0} e^{-(s-1)t} dt = \int_{0}^{1} e^{(s-1)\tau} d\tau = \frac{e^{(s-1)} - 1}{s-1}$$

with ROC the whole plane (pole at $s = 1$ is cancelled by zero at 1).
The Laplace transform of $y_c(t)$ is $Y_c(s) = 1/(s+1)$, so that

$$\begin{aligned} Y(s) &= Y_c(s) + Y_{ac}(s) = \frac{1}{s+1} + \frac{e^{s-1} - 1}{s-1} \\ &= \frac{se^{s-1} + e^{s-1} - 2}{s^2 - 1} \qquad \text{ROC } \sigma > -1 \end{aligned}$$

The poles of $Y(s)$ are $s = \pm 1$, and a zero is $s = 1$. Thus the zero cancels the pole at $s = 1$ leaving the pole at $s = -1$, and so the ROC is $\sigma > -1$.

3.9 (a) Writing

$$Y_1(s) = \frac{e^{-2s}}{s^2+1} + \frac{1}{s+2} + \frac{2}{(s+2)^3}$$

then from the table of Laplace transforms:

$$y_1(t) = \sin(t-2)u(t-2) + e^{-2t}u(t) + t^2e^{-2t}u(t)$$

(b) Expressing $Y_2(s)$ in terms $X_2(s)$ and $I(s)$ (due to the initial conditions) as:

$$\begin{aligned} Y_2(s) &= \frac{X(s)}{s^2+3s+2} + \frac{I(s)}{s^2+3s+2} \\ &= \frac{1}{s(s^2+3s+2)} + \frac{-s-1}{s^2+3s+2} \end{aligned}$$

we have that $I(s) = -s-1$, and $(s^2+3s+2)Y_2(s) = X(s)$ would give the ordinary differential equation

$$\frac{d^2y_2(t)}{dt^2} + 3\frac{dy_2(t)}{dt} + 2y_2(t) = x_2(t)$$

Its Laplace transform gives

$$(s^2+3s+2)Y_2(s) - sy_2(0) - dy_2(0)/dt - 3y_2(0) = X(s)$$

so that $I(s) = sy_2(0) + (dy_2(0)/dt + 3y_2(0))$ which compared with $I(s) = -s-1$ gives $y_2(0) = -1$ and $dy_2(0)/dt = -1 - 3y_2(0) = 2$.

(c) We have that

$$Y(s) = \frac{A}{s} + \frac{Bs+C}{(s+1)^2+4}$$

$A = Y(s)s|_{s=0} = 1/5$, then

$$Y(s) - \frac{A}{s} = \frac{-s/5 - 2/5}{(s+1)^2+4} = \frac{Bs+C}{(s+1)^2+4}$$

gives $B = -1/5$ and $C = -2/5$, so

$$Y(s) = \frac{1}{5s} - \frac{1}{5}\frac{s+1}{(s+1)^2+4} - \frac{1}{10}\frac{2}{(s+1)^2+4}$$

The steady state response is $y_{ss} = 1/5$, while the transient is

$$y_t(t) = [-(1/5)e^{-t}\cos(2t) - (1/10)e^{-t}\sin(2t)]u(t).$$

3.10 (a) Considering zero initial conditions the Laplace transform of the ordinary differential equation is

$$(s^2 + 3s + 2)Y(s) = X(s) \Rightarrow$$
$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 + 3s + 2}$$

$s^2 + 3s + 2 = (s + 1)(s + 2)$, so the poles are at $s = -1$ and $s = -2$, both in the left-hand s-plane so the system is BIBO stable. Equivalently, the impulse response $h(t) = \mathcal{L}^{-1}[H(s)]$ can be shown to be absolutely integrable. Indeed,

$$h(t) = \mathcal{L}^{-1}\left[\frac{1}{s+1} + \frac{-1}{s+2}\right] = [e^{-t} - e^{-2t}]u(t) \text{ and}$$
$$\int_{-\infty}^{\infty} |h(t)|dt = -e^{-t} + 0.5e^{-2t}|_0^{\infty} = 1 - 0.5 = 0.5$$

(b) If $X(s) = 1/s$ then

$$Y(s) = \frac{1}{s(s^2 + 3s + 2)} + \frac{I(s)}{s^2 + 3s + 2}$$

where the term $I(s)$ corresponds to the initial conditions. If $I(s) = 0$ then

$$Y(s) = \frac{1}{s(s^2 + 3s + 2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}, \text{ where } A = Y(s)s|_{s=0} = 0.5$$

and the steady state is $y_{ss}(t) = 0.5$ since the other terms disappear. If the initial conditions are not zero, the inverse Laplace transform of $I(s)/((s + 1)(s + 2))$ will give terms that disappear as $t \to \infty$ because the poles are in the left-hand s-plane. Thus the ICs do not have any effect in this case when the system is BIBO stable.

3.11 (a) The following term corresponds to the zero-state

$$\frac{(s-1)X(s)}{(s+2)^2+1}$$

and the following term corresponds to zero-input

$$\frac{1}{(s+2)^2+1}.$$

Zero-state response for $x(t)=u(t)$

$$\begin{aligned} Y_{zs}(s) &= \frac{s-1}{s((s+2)^2+1)} = \frac{A}{s} + \frac{B+Cs}{(s+2)^2+1} \\ &= \frac{-1/5}{s} + \frac{9/5+s/5}{(s+2)^2+1} = \frac{-1/5}{s} + \frac{1}{5}\,\frac{(s+2)+7}{(s+2)^2+1} \\ &= \frac{-1/5}{s} + \frac{1}{5}\,\frac{(s+2)}{(s+2)^2+1} + \frac{7}{5}\,\frac{1}{(s+2)^2+1} \end{aligned}$$

From the table of Laplace transforms

$$\begin{aligned} y_{zs}(t) &= -(1/5)u(t) + (1/5)e^{-2t}\cos(t)u(t) \\ &\quad +(7/5)e^{-2t}\sin(t)u(t) \end{aligned}$$

(b) Zero-input response

$$Y_{zi}(s) = \frac{1}{(s+2)^2+1} \Rightarrow \quad y_{zi}(t) = e^{-2t}\sin(t)u(t)$$

(c) As $t \to \infty$ $y(t) \to -1/5$.

3.12 (a) The Laplace transform of $s(t) = (h * x)(t)$ is $S(s) = H(s)X(s)$ which gives

$$S(s) = \frac{1}{(s+1)^2 + 1} \Rightarrow \quad s(t) = e^{-t}\sin(t)u(t)$$

(b) By LTI

$$y_1(t) = s(t) - s(t-1)$$
$$y_2(t) = ds(t)/dt - ds(t-1)/dt$$
$$y_3(t) = \int_{-\infty}^{t} s(t')dt'$$

3.13 (a) Zeros: $s = \pm j2$, poles: $s_1 = 0$, $s_{2,3} = -1 \pm j$. The system is not BIBO because of the pole at zero.

(b) For $x(t) = \cos(2t)u(t)$

$$Y(s) = \frac{s^2+4}{s((s+1)^2+1)} \; \frac{s}{s^2+4} = \frac{1}{(s+1)^2+1}$$

$y(t) = e^{-t}\cos(t)u(t)$, and $\lim_{t\to\infty} y(t) = 0$.

(c) When $x(t) = \sin(2t)u(t)$

$$\begin{aligned} Y(s) &= \frac{s^2+4}{s((s+1)^2+1)} \; \frac{2}{s^2+4} = \frac{2}{s((s+1)^2+1)} \\ &= \frac{1}{s} + \frac{B + C(s+1)}{(s+1)^2+1} = \frac{1}{s} + \frac{B}{(s+1)^2+1} + \frac{C(s+1)}{(s+1)^2+1} \end{aligned}$$

giving a steady state $y_{ss}(t) = 1$.
Although not needed to find the steady state, the values of B are obtained as follows

$$\frac{1}{s}\left[\frac{2}{(s+1)^2+1} - 1\right] = \frac{-s-2}{(s+1)^2+1} = \frac{B + C(s+1)}{(s+1)^2+1}$$

so that $B = C = -1$.

(d) If $x(t) = u(t)$ then

$$Y(s) = \frac{s^2+4}{s^2((s+1)^2+1)}$$

the response is a ramp and exponentially decaying sinusoids, so the steady state will be infinity or no steady–state response.

(e) The pole/zero cancellations in the first two cases permit us to find bounded responses for this unstable system. However when the input is $u(t)$ there is no pole/zero cancellation and the output is unbounded.

3.18 (a) Poles satisfy the identity $e^{-s} = 1 = e^{j2\pi k}$ for $k = 0, \pm 1, ...$, so there are an infinite number of poles, $s_k = -j2\pi k$. No zeros. The system is unstable because of poles on $j\Omega$-axis.

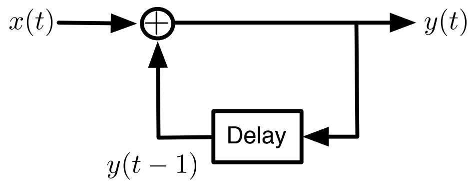

Figure 3.3: Problem 18

(b) $Y(s)(1 - e^{-s}) = X(s)$ gives $y(t) = y(t-1) + x(t)$. A positive feedback system with a delay in the feedback represents the system. See Fig. 3.3.

(c) The transfer function is

$$H(s) = \frac{1}{1 - e^{-s}} = \sum_{n=0}^{\infty} (e^{-s})^n$$

with inverse

$$h(t) = \sum_{n=0}^{\infty} \delta(t - n)$$

which is not absolutely integrable

$$\int_{-\infty}^{\infty} |h(t)| dt = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \delta(t - n) dt = \sum_{n=0}^{\infty} 1 \to \infty$$

3.20 (a) Letting $x(t) = \delta(t)$ so that $y(t) = h(t)$ we get

$$h(t) = \alpha\delta(t-T) + \alpha^3\delta(t-3T)$$

(b) $H(s) = \alpha e^{-sT} + \alpha^3 e^{-s3T}$
$H(s)$ has no poles, its zeros satisfy $\alpha e^{-sT} + \alpha^3 e^{-s3T} = 0$ or (dividing by $\alpha e^{-sT} > 0$)

$$\begin{aligned} 1 &= e^{-j\pi}\alpha^2 e^{-2Ts} = \alpha^2 e^{-2T(\sigma + j(\Omega+\pi/(2T))}\ \text{ letting }\ s = \sigma + j\Omega \\ 1e^{-j2\pi k} &= (\alpha^2 e^{-2T\sigma})e^{-j2T(\Omega+\pi/(2T))},\ \ k = 0, \pm 1, \pm 2, \cdots \end{aligned}$$

so that

$$\begin{aligned} & 1 = \alpha^2 e^{-2T\sigma} \Rightarrow \ \ \sigma = \frac{\log\alpha^2}{2T},\ \text{ real part of zeros} \\ & -j2T(\Omega + \pi/(2T)) = -jk2\pi \Rightarrow \ \ \Omega = \frac{(2k-1)\pi}{2T},\ \text{ imaginary part of zeros} \\ & \text{zeros}\ \ \ s = \sigma + j\Omega = \frac{\log\alpha^2}{2T} + j\left(\frac{(2k-1)\pi}{2T}\right) \end{aligned}$$

System is BIBO stable since $\sigma = \log(\alpha)/T < 0$ because $0 < \alpha < 1$.

3.22 (a)

$$X(s) = \frac{1}{s}(1 - 2e^{-s} + e^{-2s}) = \frac{1}{s}(1 - e^{-s})^2$$

with the whole s-plane as ROC. Pole $s = 0$ is cancelled by zero $s = 0$. The transfer function is

$$H(s) = \frac{Y(s)}{X(s} = \frac{s+2}{(s+1)^2} = \frac{1}{s+1} + \frac{1}{(s+1)^2} \qquad \text{ROC } \sigma > 0$$

Using the frequency shift property, the inverse of $1/(s+1)^2$ is $e^{-t}r(t)$, so

$$h(t) = e^{-t}(1+t)u(t)$$

(b) The partial fraction expansion gives

$$X(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 10}$$

Since the roots of $s^2 + 2s + 10 = (s+1)^2 + 9 = 0$ are

$$s_{1,2} = -1 \pm j3$$

the steady state is given by the A/s term. As $t \to \infty$, the terms corresponding to complex roots goes to zero. We have $A = 1/10$ and

$$\lim_{t \to \infty} x(t) = 1/10$$

(c) The poles of $Z(s)$ are $s = 0$ and $s = -2 \pm j1$. The inverse transform corresponding to the complex roots goes to zero in the steady state. Thus

$$\lim_{t \to \infty} z(t) = sZ(s)|_{s=0} = 1/5$$

(d) Since e^{-s} simply shifts the inverse by one, it does not affect the steady state. Letting $W_1(s) = 1/(s((s-2)^2 + 1)$ the complex poles $s_{12} = 2 \pm j$ have a positive real part so that its response will grow as t increases and as such there will be no steady–state.

(e) The partial fraction expansion is

$$V(s) = \frac{A}{s} + \frac{B + Cs}{(s+1)^2 + 1}$$

where

$$A = V(s)s|_{s=0} = 0.5$$

so that the steady state is 0.5 since the other term has poles in the left-hand s-plane, $s_{1,2} = -1 \pm j1$, which would give a transient. If we subtract $0.5/s$ from $V(s)$ we obtain the Laplace transform of the transient, thus

$$V(s) - \frac{0.5}{s} = -\frac{0.5s}{(s+1)^2 + 1} = -\frac{0.5s + 0.5}{(s+1)^2 + 1} + \frac{0.5}{(s+1)^2 + 1}$$

which gives the transient

$$v_t(t) = -0.5e^{-t}\cos(t)u(t) + 0.5e^{-t}\sin(t)u(t)$$

3.25 (a) The Laplace transform of the output for input $x(t)$ is

$$Y(s) = H_1(s)X(s) = \frac{X(s)}{(s+1)(s-2)} = \cdots + \frac{B}{s+1} + \frac{C}{s-2}$$

where the dots correspond to the partial fraction expansion of the poles of $X(s)$, and we assume that there is no pole/zero cancellation. Then

$$\lim_{t\to\infty} y(t) \to \infty$$

because the system is unstable due to the $s = 2$ pole in the right-hand s-plane, as it would have an inverse $Ce^{2t}u(t)$ which grows as time increases.
(b) In this case, the Laplace transform of the output for any bounded input $x(t)$ is

$$Y(s) = H_2(s)X(s) = \frac{X(s)}{(s+1)(s+2)} = \cdots + \frac{B}{s+1} + \frac{C}{s+2}$$

then assuming no pole/zero cancellation

$$\lim_{t\to\infty} y(t) \to 0$$

because the system is stable (poles in the left s-plane), and all the poles of $Y(s)$ are in the left-hand s-plane. All terms have as inverse a signal that decays as t increases.

Since the input is bounded, the output should also be bounded for the system to be BIBO stable, but we only know that the steady state goes to zero. So the fact that $y(t)$ tends to zero simply tells that there is the possibility that the system is not unstable, unfortunately it does not imply the system is stable. The condition for stability is that the impulse response of the system be absolutely integrable or that the poles of the transfer function be in the left-hand s-plane, which in this case are, so the system is BIBO stable.

3.29 (a) Negative feedback

$$\begin{aligned} G(s) &= \frac{Y(s)}{C(s)} \\ Y(s) &= H(s)(C(s) - Y(s)) \\ G(s) &= \frac{H(s)}{1 + H(s)} \end{aligned}$$

(b) Positive feedback

$$\begin{aligned} G_1(s) &= \frac{Y(s)}{C(s)} \\ Y(s) &= H(s)(C(s) + Y(s)) \\ G_1(s) &= \frac{H(s)}{1 - H(s)} \end{aligned}$$

(c) Negative feedback gives

$$G(s) = \frac{1/(s+1)}{1 + 1/(s+1)} = \frac{1}{s+2}$$

while positive feedback gives

$$G_1(s) = \frac{1/(s+1)}{1 - 1/(s+1)} = \frac{1}{s}$$

The Laplace transform of the output when $C(s) = 1/s$ is for negative feedback

$$Y(s) = G(s)C(s) = \frac{1}{s(s+2)} = \frac{0.5}{s} - \frac{0.5}{s+2}$$

so that $y(t) = 0.5u(t) - 0.5e^{-2t}u(t)$.
For the positive feedback

$$Y(s) = G_1(s)C(s) = \frac{1}{s^2}$$

so that $y(t) = r(t)$ which will go to ∞ as $t \to \infty$.

3.30 (a) The transfer function of the overall system is

$$G(s) = \frac{H(s)}{1 + KH(s)} = \frac{2}{s + 2K - 1}$$

for the pole to be in the left-hand s-plane we need $2K - 1 > 0$ or $K > 0.5$, for instance $K = 1$ which gives a pole at $s = -1$ so that the impulse response of the feedback system is $g(t) = 2e^{-t}u(t)$ which is absolutely integrable.

(b) We have

i. The transfer function of an all-pass filter is of the form

$$H_{ap}(s) = K \prod_k \frac{s - \sigma_k}{s + \sigma_k} \prod_m \frac{(s - \sigma_m + j\Omega_m)(s - \sigma_m - j\Omega_m)}{(s + \sigma_m + j\Omega_m)(s + \sigma_m - j\Omega_m)}$$

for $\sigma_k,\ \sigma_m > 0$.

ii. The all-pass is

$$H_{ap}(s) = K\frac{s - 1}{s + 1}$$

iii. To cancel the pole at $s = 1$, so that the gain of the all-pass be unity the gain at $s = 0$ should be unity, i.e.,

$$H_{ap}(0) = K(-1) = 1$$

so that $K = -1$ and the filter becomes

$$H_{ap}(s) = \frac{1 - s}{s + 1}$$

giving a serial connection with a transfer function

$$G(s) = H(s)H_{ap}(s) = \frac{s + 1}{(s - 1)(s^2 + 2s + 1)} \frac{-(s - 1)}{s + 1} = \frac{-1}{(s + 1)^2}$$