# Chapter 2

# Continuous–time Systems

## 2.1 Basic Problems

**2.1** (a) The $y(t)$-$x(t)$ relation is a line through the origin between $-10$ to $10$ and a constant before and after that. The system is non-linear, for instance if $x(t) = 7$ the output is $y(t) = 700$ but if we double the input, the output is not $2y(t) = 1400$ but 1000.
(b) If the inputs is always between $-10$ and $10$ the system behaves like a linear system. In this case the output is chopped whenever $x(t)$ is above $10$ or below $-10$. Se Fig. 2.1.
(c) Whenever the input goes below $-10$ or above $10$ the output is $-1000$ and $1000$, otherwise the output is $2000\cos(2\pi t)u(t)$.
(d) If the input is delayed by 2 the clipping will still occur, simply at a later time. So the system is time invariant.

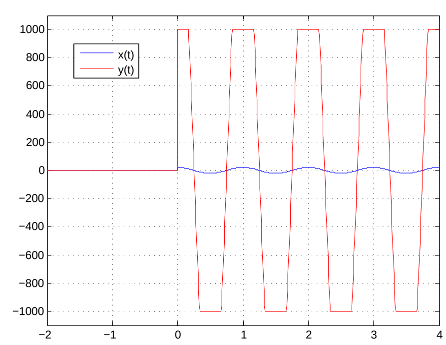


Figure 2.1: Problem 1: input and output of amplifier.

**2.2** (a) Input $x_1(t) = \delta(t)$ gives

$$y_1(t) = \int_{t-1}^{t} \delta(\tau)d\tau + 2 = \begin{cases} 2 & t < 0 \\ 3 & 0 \leq t \leq 1 \\ 2 & t > 1 \end{cases}$$

$x_2(t) = 2x_1(t)$ gives

$$y_2(t) = 2\int_{t-1}^{t} \delta(\tau)d\tau + 2 = \begin{cases} 2 & t < 0 \\ 4 & 0 \leq t \leq 1 \\ 2 & t > 1 \end{cases}$$

Since $y_2(t) \neq 2y(t)$ system is non-linear.

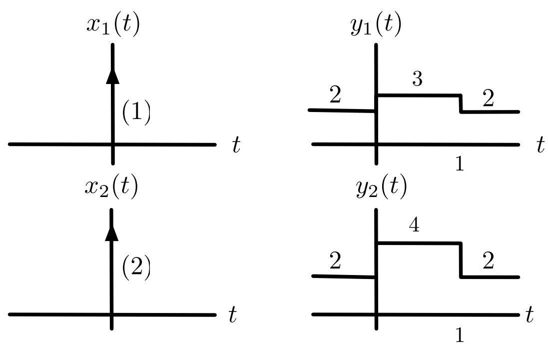


Figure 2.2: Problem 2

(b) If $x_3(t) = u(t) - u(t-1)$ then $y_3(t) = 2 + r(t) - 2r(t-1) + r(t-2)$ . If $x_4(t) = x_3(t-1)$ then the corresponding output is $y_3(t-1)$, so the system is time-invariant.

(c) Non-causal, although $y(t)$ depends on present and past inputs, it is not zero when $x(t) = 0$, due to the bias of 2.

(d) If $|x(t)| < M$ we have

$$|y(t)| \leq \int_{t-1}^{t} |x(\tau)|d\tau + 2 < M + 2 < \infty$$

The system is BIBO stable.

**2.4** (a) Derivative

$$\frac{dz(t)}{dt} = w(t) - w(t-1)$$

which excludes the initial condition of 2. System is LTI if initial condition is zero.

(b) i. If input is $i(t-\mu)$ then the output is letting $\eta = \tau - \mu$

$$\int_0^t i(\tau - \mu)d\tau = \int_{-\mu}^0 i(\eta)d\eta + \int_0^{t-\mu} i(\eta)d\eta = v_c(t-\mu)$$

that is, provided that $i(t) = 0$ for $t < 0$, the system is time-invariant.

ii. If $i(t) = u(t)$ then $v_c(t) = \int_0^t u(\tau)d\tau = r(t)$. If we shift the inputs $i_1(t) = i(t-1) = u(t-1)$ the previous output is shifted, so system is time-invariant.

(c) If $x(t) = u(t)$ then $y(t) = \sin(2\pi t)u(t)$ while corresponding to $x(t-0.5) = u(t-0.5)$ is $y_1(t) = \sin(2\pi t)u(t-0.5)$ indicating the system is not time-invariant as $y_1(t)$ is not $y(t-0.5)$.

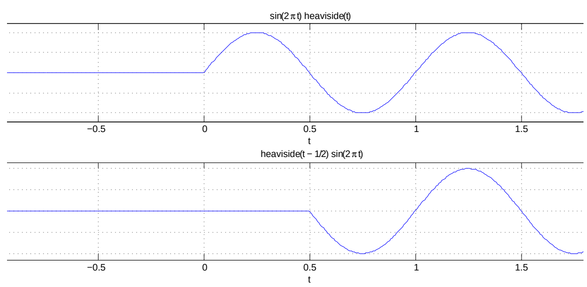


Figure 2.3: Problem 4(c)

**2.5** (a) See Fig. 1. The circuit is a series connection of a voltage source $x(t)$ with a resistor $R = 1/2\ \Omega$, and capacitor $C = 1$F. Indeed, the mesh current is $i(t) = dy(t)/dt$ so

$$x(t) = Ri(t) + y(t) = Rdy(t)/dt + y(t)$$

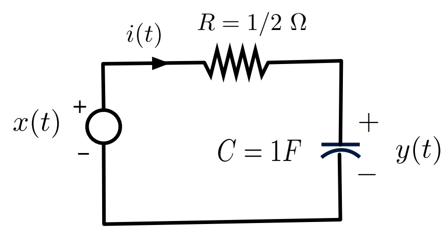


Figure 2.4: Problem 5

(b) The output is

$$y(t) = e^{-2t}\, 0.5\, e^{2\tau}|_0^t = 0.5(1 - e^{-2t})u(t)$$

and

$$\begin{aligned} \frac{dy(t)}{dt} &= e^{-2t}u(t) + 0.5(1 - e^{-2t})\delta(t) \\ &= e^{-2t}u(t) \\ \frac{dy(t)}{dt} + 2y(t) &= e^{-2t}u(t) + u(t) - e^{-2t}u(t) \\ &= u(t) \end{aligned}$$

**2.7** (a) The charge is

$$q(t) = C(t)v(t)$$

so that

$$i(t) = \frac{dq(t)}{dt} = C(t)\frac{dv(t)}{dt} + v(t)\frac{dC(t)}{dt}$$

(b) If $C(t) = 1 + \cos(2\pi t)$ and $v(t) = \cos(2\pi t)$, the current is

$$\begin{aligned} i_1(t) &= C(t)\frac{dv(t)}{dt} + v(t)\frac{dC(t)}{dt} \\ &= (1 + \cos(2\pi t))(-2\pi \sin(2\pi t)) - \cos(2\pi t)(2\pi \sin(2\pi t)) \\ &= -2\pi \sin(2\pi t)[1 + 2\cos(2\pi t)] \end{aligned}$$

(c) When the input is

$$v(t - 0.25) = \cos(2\pi(t - 1/4)) = \sin(2\pi t)$$

the output current is

$$\begin{aligned} i_2(t) &= C(t)\frac{dv(t-0.25)}{dt} + v(t-0.25)\frac{dC(t)}{dt} \\ &= (1 + \cos(2\pi t))(2\pi \cos(2\pi t)) - 2\pi \sin^2(2\pi t) \\ &= 2\pi \cos(2\pi t) + 2\pi[\cos^2(2\pi t) - \sin^2(2\pi t)] \end{aligned}$$

which is not

$$i_1(t - 0.25) = 2\pi \cos(2\pi t)[1 + \sin(2\pi t)]$$

so the system is time varying.

**2.8** (a) The system is LTI since the input $x(t)$ and the output $y(t)$ are related by a convolution integral with $h(t-\tau) = e^{-(t-\tau)}u(t-\tau)$ or $h(t) = e^{-t}u(t)$.
Another way: to show that the system is linear let the input be $x_1(t) + x_2(t)$, and $x_1(t)$ and $x_2(t)$ have as outputs

$$y_i(t) = \int_0^t e^{-(t-\tau)} x_i(\tau) d\tau \qquad i = 1, 2$$

The output for $x_1(t) + x_2(t)$ is

$$\int_0^t e^{-(t-\tau)}(x_1(\tau) + x_2(\tau))d\tau = y_1(t) + y_2(t)$$

To show the time invariance let the input be $x(t-t_0)$, its output will be

$$\begin{aligned} \int_0^t e^{-(t-\tau)} x(\tau - t_0) d\tau &= \int_{-t_0}^0 e^{-((t-t_0)-\mu)} x(\mu) d\mu + \int_0^{t-t_0} e^{-((t-t_0)-\mu)} x(\mu) d\mu \\ &= \int_0^{t-t_0} e^{-((t-t_0)-\mu)} x(\mu) d\mu = y(t-t_0) \end{aligned}$$

by letting $\mu = \tau - t_0$ and using the causality of the input. The system is then TI.

Finally the impulse response is found by letting $x(t) = \delta(t)$ so that the output is

$$h(t) = \int_0^t e^{-(t-\tau)}\delta(\tau)d\tau = \int_0^t e^{-(t-0)}\delta(\tau)d\tau = \begin{cases} e^{-t} \times 1 = e^{-t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) Yes, this system is causal as the output $y(t)$ depends on present and past values of the input.
(c) Letting $x(t) = u(t)$, the unit-step response is

$$s(t) = \int_0^t e^{-t+\tau} u(\tau) d\tau = e^{-t} \int_0^t e^{\tau} d\tau = 1 - e^{-t}$$

for $t \geq 0$ and zero otherwise. The impulse response as indicated before is $h(t) = ds(t)/dt = e^{-t}u(t)$. The BIBO stability of the system is then determined by checking whether the impulse response is absolutely integrable or not,

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_0^{\infty} e^{-t} dt = -e^{-t}\,|_0^{\infty} \, = 1$$

so yes it is BIBO stable.
(d) Using superposition, the response to the pulse $x_1(t) = u(t) - u(t-1)$ would be

$$y_1(t) = y(t) - y(t-1) = (1 - e^{-t})u(t) - (1 - e^{-(t-1)})u(t-1)$$

which starts at zero, grows to a maximum of $1 - e^{-1}$ at $t = 1$ and goes down to zero as $t \to \infty$.

**2.9** (a) Letting $x(t) = \delta(t)$ the impulse response is

$$\begin{aligned} h(t) &= \frac{1}{T}\int_{t-T/2}^{t+T/2} \delta(\tau)d\tau \\ &= \frac{1}{T}\int_{t-T/2}^{t} \delta(\tau)d\tau + \frac{1}{T}\int_{t}^{t+T/2} \delta(\tau)d\tau \end{aligned}$$

If $t > 0$, and $t - T/2 < 0$ the first integral includes $0$, while the second does not. Thus

$$h(t) = \frac{1}{T}\int_{t-T/2}^{t} \delta(\tau)d\tau + 0 = \frac{1}{T} \qquad t > 0 \text{ and } t - T/2 < 0, \text{ or } \; 0 < t < T/2$$

Likewise when $t < 0$ then $t - T/2 < -T/2$ and $t + T/2 < T/2$ the reverse of the previous case happens and so

$$h(t) = 0 + \frac{1}{T}\int_{t}^{t+T/2} \delta(\tau)d\tau = \frac{1}{T} \qquad t < 0 \text{ and } t + T/2 > 0, \text{ or } \; -T/2 < t < 0$$

so that

$$h(t) = \frac{1}{T}[u(t + T/2) - u(t - T/2)]$$

indicating that the system is non-causal as $h(t) \neq 0$ for $t < 0$.
(b) If $x(t) = u(t)$ then the output of the averager is

$$y(t) = \frac{1}{T}\int_{t-T/2}^{t+T/2} u(\tau)d\tau$$

If $t + T/2 < 0$ then $y(t) = 0$ since the argument of the unit step signal is negative. If $t + T/2 \geq 0$ and $t - T/2 < 0$ then

$$y(t) = \int_{0}^{t+T/2} u(\tau)d\tau = \frac{1}{T}(t + T/2)$$

and finally when $t - T/2 \geq 0$ then

$$y(t) = \frac{1}{T}\int_{t-T/2}^{t+T/2} u(\tau)d\tau = 1$$

The unit-step response of the noncausal averager is

$$y(t) = \begin{cases} 0 & t < -T/2 \\ \frac{1}{T}(t + T/2) & -T/2 \leq t < T/2 \\ 1 & t \geq T/2 \end{cases}$$

**2.12** (a) If $y(0) = 0$ the system is linear, indeed for an input $\alpha x_1(t) + \beta x_2(t)$ with $y_1(t)$ the response due to $x_1(t)$ and $y_2(t)$ the response due to $x_2(t)$ we have

$$\int_0^t e^{-(t-\tau)}[\alpha a x_1(\tau) + \beta x_2(\tau)]d\tau = \alpha y_1(t) + \beta y_2(t)$$

If $y(0) \neq 0$, the output for input $\alpha x_1(t)$ is

$$y(0)e^{-t} + \int_0^t e^{-(t-\tau)}\alpha x_1(\tau)d\tau = y(0)e^{-t} + \alpha y_1(t)$$

which is not $\alpha y_1(t)$ thus it is not linear.

(b) If the input is $x(t) = 0$, then $y(t) = y(0)e^{-t}u(t)$ is the zero-input response, due completely to the initial condition. If $y(0) = 0$ the response

$$y(t) = \int_0^t e^{-(t-\tau)}x(\tau)d\tau$$

(which is the convolution integral of the impulse response $h(t) = e^{-t}u(t)$ with $x(t)$) is the zero-state response.

(c) The impulse response, obtained when $y(0) = 0$, $x(t) = \delta(t)$, and $y(t) = h(t)$ is

$$h(t) = \int_{0-}^t e^{-(t-\tau)}\delta(\tau)d\tau = e^{-t}\int_{0-}^t e^0\delta(\tau)d\tau = \begin{cases} e^{-t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(d) If $x(t) = u(t)$ and $y(0) = 0$, then $y(t) = s(t)$ given by

$$s(t) = \int_0^t e^{-(t-\tau)}d\tau = (1 - e^{-t})u(t)$$

Notice the relation between the unit-step and the impulse response:

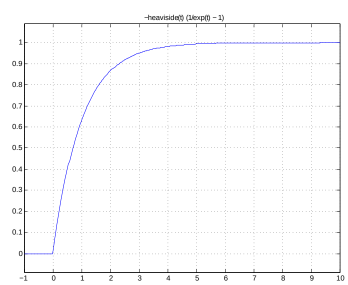


Figure 2.5: Problem 12

$$\begin{aligned} \frac{ds(t)}{dt} &= \delta(t) - e^{-t}\delta(t) + e^{-t}u(t) \\ &= e^{-t}u(t) = h(t) \end{aligned}$$

**2.14** (a) Yes. Using the convolution integral the output is

$$y(t) \quad = \quad \int_{-\infty}^{\infty} \underbrace{h(\tau)}_{u(\tau)-u(\tau-1)} x(t-\tau)d\tau = \int_0^1 x(t-\tau)d\tau = \int_{t-1}^{t} x(\eta)d\eta$$

where we changed the variable to $\eta = t - \tau$.

(b) If $x(t) = u(t)$ then the step-response is

$$y(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 1 & t \geq 1. \end{cases}$$

i.e., the unit-step response is $s(t) = r(t) - r(t-1)$ and the impulse response is

$$h(t) = \frac{ds(t)}{dt} = u(t) - u(t-1)$$

**2.15** (a) $x_1(t) = x(t) - x(t-2)$ so $y_1(t) = y(t) - y(t-2)$, two triangular pulses, the second multiplied by $-1$.

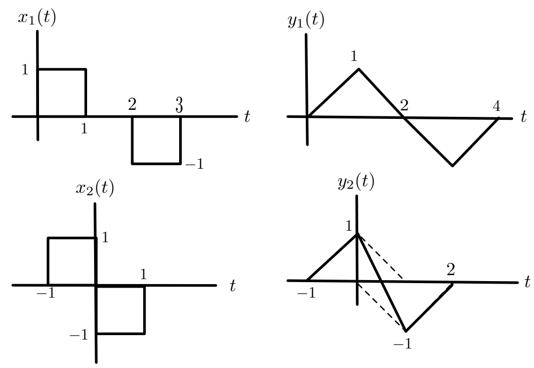


Figure 2.6: Problem 15

(b) $x_2(t) = x(t+1) - x(t)$ then $y_2(t) = y(t+1) - y(t)$ (they overlap between 0 and 1).

(c) $x_3(t) = \delta(t) - \delta(t-1)$ so $y_3(t) = dy(t)/dt = u(t) - 2u(t-1) + u(t-2)$. Considering that the output of $x(t)$ is $y(t)$, i.e., $y(t) = \mathcal{S}[x(t)]$, and that the integrator and the differentiator are LTI systems Fig. 2.7 shows how to visualize the result in this problem by considering that you can change the order of the cascading of LTI systems.

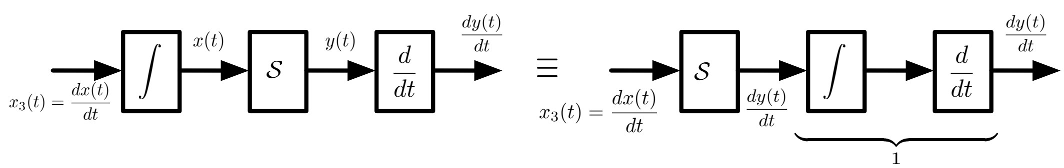


Figure 2.7: Problem 15

**2.18** (a) For $x_1(t) = u(t) - u(t-2)$ and zero initial conditions, we can use the convolution integral

$$\begin{aligned} y_1(t) &= \int_0^t h(\tau)x_1(t-\tau)d\tau \\ &= \sum_{k=0}^{\infty} \int_0^t h_1(\tau - 2k)x(t-\tau)d\tau \end{aligned}$$

for $k = 0$, we find graphically the integral to be

$$z(t) = \int_0^t h_1(\tau)x_1(t-\tau)d\tau = r(t) - 2r(t-1) + 2r(t-3) - r(t-4)$$

graphically. So that

$$y_1(t) = \sum_{k=0}^{\infty} z(t-2k) = r(t) - 2r(t-1) + r(t-2)$$

(b) For $x_2(t) = \delta(t) - \delta(t-2)$, the output is

$$\begin{aligned} y_2(t) &= h(t) - h(t-2) \\ &= \sum_{k=0}^{\infty} h_1(t-2k) - \sum_{k=0}^{\infty} h_1(t-2(k+1)) \\ &= h_1(t) + \sum_{k=1}^{\infty} h_1(t-2k) - \sum_{k'=1}^{\infty} h_1(t-2k') \\ &= h_1(t) \end{aligned}$$

where we changed to the variable $k' = k+1$ in the second summation.
Also since $x_2(t) = dx_1(t)/dt$ the output

$$y_2(t) = \frac{dy_1(t)}{dt} = u(t) - 2u(t-1) + u(t-2) = h_1(t)$$