# Chapter 11

# Fourier Analysis of Discrete–time Signals and Systems

## 11.1 Basic Problems

**11.1** The DTFT of $x[n] = 0.5^{|n|}$ is

$$X(e^{j\omega}) = \frac{3/4}{5/4 - \cos(\omega)}$$

(a) If we let $\omega = 0$ then

$$X(1) = \frac{3/4}{5/4 - 1} = 3 = \sum_n x[n]$$

(b) The inverse DTFT is

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

if we let $n = 0$ we get that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) d\omega = x[0]$$

and so the given integral is $2\pi x[0] = 2\pi$.
(c) From the DTFT, $X(e^{j\omega})$ is real and since the denominator, i.e., $5/4 - \cos(\omega)$, is positive for $[-\pi, \pi)$ the phase $\angle X(e^{j\omega}) = 0$.
(d) If we let $\omega = \pi$ in the DTFT we obtain

$$\sum_n x[n](-1)^n = X(e^{j\pi n}) = \frac{3/4}{9/4} = \frac{1}{3}$$

**11.2** (a) We have

i. DTFT

$$X(e^{j\omega}) = \sum_{n=-2}^{2} e^{-j\omega n} = 1 + 2\cos(\omega) + 2\cos(2\omega)$$

ii. Z-transform

$$X(z) = z^2 + z + 1 + z^{-1} + z^{-2} \qquad \text{ROC } |z| > 0$$
$$X(e^{j\omega}) = X(z)|_{z=e^{j\omega}} \qquad \text{ROC includes UC}$$

which coincides with the previous result.

iii. $X(e^{j0}) = \sum_{n=-2}^{2} 1 = 5$

(b) Problem 2(b)
We have

$$X(z) = \sum_{n=-\infty}^{0} \alpha^n z^{-n} = \sum_{m=0}^{\infty} \alpha^{-m} z^m = \frac{1}{1 - z/\alpha}, \;\; \text{ROC: } |z| < \alpha$$

so ROC must include the unit circle (UC) and as such $\alpha > 1$.

(c) Problem 2(c)

i. Z-transforms

$$\begin{aligned} X_1(z) &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \frac{1}{1 - 0.5z^{-1}} = \frac{z}{z - 0.5} \qquad |z| > 0.5 \\ X_2(z) &= -\sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^n z^{-n} = -\sum_{m=0}^{\infty} 2^m z^m + 1 = \frac{z}{z - 0.5} \qquad |z| < 0.5 \end{aligned}$$

ii. Since the ROC $X_1(z)$ includes the UC then $X_1(e^{j\omega}) = X_1(z)|_{z=e^{j\omega}}$.

**11.3** Writing

$$t[n] = \sum_{k=-2}^{2}(3-|k|)\delta[n-k] = 3\delta[n] + \sum_{k=1}^{2}(3-k)(\delta[n+k]+\delta[n-k])$$

$A_k = 3-|k|$, for $-2 \leq k \leq 2$, 0 otherwise.
The Z-transform of $t[n]$ is

$$T(z) = 3 + \sum_{k=1}^{2}(3-k)(z^k + z^{-k})$$

so that the DTFT is

$$\begin{aligned} T(e^{j\omega}) &= 3 + \sum_{k=1}^{2}(3-k)[e^{j\omega k} + e^{-j\omega k}] \\ &= \underbrace{3}_{B_0} + \sum_{k=1}^{2}\underbrace{2(3-k)}_{B_k}\cos(k\omega) \end{aligned}$$

for $k > 2$, $B_k = 0$.

**11.4** (a) Impulse response

$$h[n] = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{j\omega n} d\omega = \begin{cases} 0.5 & n = 0 \\ \sin(\pi n/2)/(\pi n) & n \neq 0 \end{cases}$$

$h[n]$ is non-causal as $h[n] \neq 0$ for $n < 0$.

(b) $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = H(e^{j\omega})$ so $y[n] = h[n]$

(c) Yes, $H(e^{j\omega}) = H(e^{j\omega})H(e^{j\omega})$ so $h[n] = (h * h)[n]$.

**11.5** (a) The DTFT of $x[n] = e^{j\theta}\delta[n+\tau] + e^{-j\theta}\delta[n-\tau]$ is

$$X(e^{j\omega}) = e^{j\theta}e^{j\omega\tau} + e^{-j\theta}e^{-j\omega\tau} = 2\cos(\omega\tau + \theta)$$

by duality ($\omega \to n$, $\tau \to \omega_0$)

mistake (cf table): exp(-j theta) delta(omega+omega_0) +...

$$\cos(n\omega_0 + \theta) \;\leftrightarrow\; \pi\left[e^{j\theta}\delta(\omega+\omega_0) + e^{-j\theta}\delta(\omega-\omega_0)\right]$$

For $\theta = 0$

$$\cos(n\omega_0) \;\leftrightarrow\; \pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]$$

For $\theta = \pi/2$

minus-sign missing: cos(n omega_0+pi/2) = -sin(n omega_0)

$$\cos(n\omega_0 + \pi/2) = \sin(n\omega_0) \;\leftrightarrow\; \pi\left[j\delta(\omega+\omega_0) - j\delta(\omega-\omega_0)\right]$$

(b) Replacing DTFT of cosine terms

$$X_1(e^{j\omega}) \;=\; 2\pi\delta(\omega) + \sum_{k=1}^{5} A_k\pi[e^{j\theta_k}\delta(\omega+k\omega_0) + e^{-j\theta_k}\delta(\omega-k\omega_0)]$$

**11.6** (a) We have

$$h_1[n] = h[n](1+e^{j\pi n}) = h[n] + h[n]e^{j\pi n} = \begin{cases} 2h[n] & n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

so

$$H_1(e^{j\omega}) = H(e^{j\omega}) + H(e^{j(\omega+\pi)})$$

thus since $H(e^{j\omega})$ corresponds to a LPF then $H_1(e^{j\omega})$ is a band-eliminating filter. Sketch its frequency response to verify it.

(b) i. Using DTFT

$$H(e^{j0}) = \sum_{n=-\infty}^{\infty} h[n] = \frac{0.75}{1.25-1} = 3$$

ii. Using IDTFT

$$\begin{aligned} h[-n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{-j\omega n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{-j\omega})e^{-j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega'})e^{j\omega' n}d\omega' = h[n] \end{aligned}$$

where $\omega' = -\omega$.

iii. The denominator $1.25 - \cos(\omega)$ of $H(e^{j\omega})$ is positive for $(-\pi, \pi]$, so $H(e^{j\omega})$ is real and positive, with zero phase.

**11.7** (a) (a) (b) DTFT

$$X(e^{j\omega}) = 1 - e^{-j2\omega} = 2je^{-j\omega}\sin(\omega)$$

Yes, $X(e^{j\omega})$ is periodic of period $2\pi$ since $\omega + 2k\pi = \omega$. $|X(e^{j\omega})| = 2|\sin(\omega)|$ is periodic of

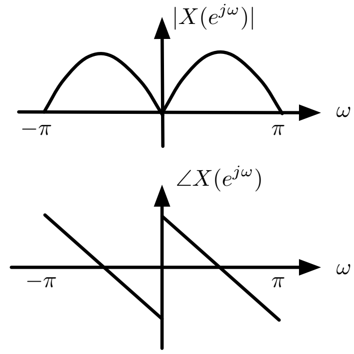


Figure 11.1: *Problem 7: Magnitude and phase of* $X(e^{j\omega})$

period $\pi$, but also periodic of period $2\pi$.

(b) Phase

$$\angle X(e^{j\omega}) = \begin{cases} -\omega + \pi/2 & \text{if } \sin(\omega) > 0 \\ -\omega + 3\pi/2 = -\omega - \pi/2 & \text{if } \sin(\omega) \leq 0 \end{cases}$$

and because of the odd symmetry of the phase it is zero at $\omega = 0$. See Fig. 11.1.

**11.9** (a) If $\hat{H}(e^{j\omega}) = A[u(\omega+\omega_0) - u(\omega-\omega_0)]$ with zero phase. To determine $A$ and $\omega_0$ find the impulse response

$$\hat{h}[n] = \frac{A}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega n} d\omega = \frac{A}{\pi n}\sin(\omega_o n)$$

so $\omega_0 = \pi/3$ and $A = 1$ and since $h[n] = \hat{h}[n-10]$ we have

$$H(e^{j\omega}) = e^{-j10\omega}\hat{H}(e^{j\omega}) = e^{-j10\omega}[u(\omega+\pi/3) - u(\omega-\pi/3)]$$

so the magnitude and phase responses are

$$|H(e^{j\omega})| = \begin{cases} 1 & -\pi/3 \le \omega \le \pi/3 \\ 0 & \text{otherwise in } -\pi < \omega \le \pi \end{cases}$$
$$\angle H(e^{j\omega}) = -10\omega \qquad -\pi < \omega \le \pi$$

(b) $X(e^{j\omega}) = e^{-j\omega} + \pi\delta(\omega+\pi/5) + \pi\delta(\omega-\pi/5)$ then

$$\begin{aligned} Y(e^{j\omega}) &= X(e^{j\omega})H(e^{j\omega}) \\ &= H(e^{j\omega})e^{-j\omega} + \pi H(e^{-j\pi/5})\delta(\omega+\pi/5) + \pi H(e^{j\pi/5})\delta(\omega-\pi/5) \\ &= H(e^{j\omega})e^{-j\omega} + \pi e^{j10\pi/5}\delta(\omega+\pi/5) + \pi e^{-j10\pi/5}\delta(\omega-\pi/5) \\ &= H(e^{j\omega})e^{-j\omega} + \pi\delta(\omega+\pi/5) + \pi\delta(\omega-\pi/5) \end{aligned}$$

and

$$y[n] = h[n-1] + \cos(\pi n/5) \qquad -\infty < n < \infty$$

**11.10** (a) The Z-transform

$$X(z) = \frac{1}{1-\beta z^{-1}} \;\; |z| > |\beta|$$

So when $\beta \geq 1$ the region of convergence does not include the unit circle so the DTFT $X(e^{j\omega})$ cannot be found. If $\beta < 1$, the ROC includes the UC and

$$X(e^{j\omega}) = X(z)|_{z=e^{j\omega}}$$

(b) We have

i. The inverse is

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\omega)e^{j\omega n}d\omega = \frac{1}{2\pi}$$

ii. The inverse is

$$x_1[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\omega-\pi)e^{j\omega n}d\omega = \frac{(-1)^n}{2\pi}$$

iii. $\delta(\omega) = \delta(-\omega)$ so $x_2[n] = 2x[n] = 1/\pi$.

(c) Yes, cross-multiplying

$$(1 + e^{-j\omega} + \cdots + e^{-j\omega(N-1)})(1 - e^{-j\omega}) = 1 - e^{-j\omega N}$$

when $N = 1$ we get an identity, $1 = (1 - e^{-j\omega})/(1 - e^{-j\omega})$.

**11.11** (a) Frequency response

$$H(e^{j\omega}) = \frac{e^{-j\omega}}{3}(1 + 2\cos(\omega))$$
$$|H(e^{j\omega})| = \left|\frac{1 + 2\cos(\omega)}{3}\right|$$
$$\angle H(e^{j\omega}) = \begin{cases} -\omega & 1 + 2\cos(\omega) \geq 0 \\ -\omega + \pi & 1 + 2\cos(\omega) < 0 \end{cases}$$

(b) Poles and zeros

$$H(z) = \frac{z^2 + z + 1}{3z^2} = \frac{(z + 0.5)^2 + 3/4}{3z^2}$$
$$\text{Poles: } z = 0 \text{ double, zeros: } z_{1,2} = -0.5 \pm j\sqrt{3}/2$$

the poles are at the origin and the zeros on the unit circle ($|z_{1,2}| = 1$. The ROC is the whole Z–plane except for $z = 0$.

(c) If $1 + 2\cos(\omega_0) = 0$, then the magnitude is zero and this happens at $\omega_0 = \cos^{-1}(-0.5)$

(d) Because the zeros are on the unit circle the difference between discontinuities in the wrapped phase is not $2\pi$, so unwrapping would not change the phase.

**11.12** (a) Frequency response

$$H(e^{j\omega}) = e^{-j2\omega}(1.2 + \cos(\omega))$$
$$\angle H(e^{j\omega}) = -2\omega \text{ linear, as } \quad 1.2 + \cos(\omega) > 0 \; \omega \in (-\pi, \pi]$$

(b) Impulse response

$$H(z) = \underbrace{0.5}_{h[1]} z^{-1} + \underbrace{1.2}_{h[2]} z^{-2} + \underbrace{0.5}_{h[3]} z^{-3}$$

$h[n]$ is symmetric with respect to $n = 2$, so the phase is linear.