Chapter 10

The Z–transform

10.1 Basic Problems

10.1 (a) For $T_s = 1$ the transformation from the s-plane to the Z–plane $z = e^s$ is such that for $s = \sigma + j\Omega$

$$z = e^{\sigma} e^{j\Omega}$$

For $s = -1 \pm j1$, the poles of the analog system, the corresponding singularities in the Z–plane are given by

$$z_{12} = e^{-1} e^{\pm j\ 1}$$

which are inside the unit disk as $e^{-1} < 1$ with a radian frequency of ± 1.
The pole $s = 0$ is mapped into $z = e^0 = 1$, and the poles $s = \pm j1$ are mapped into $z = 1e^{\pm j1}$ with unit magnitude and radian frequencies ± 1.
(b) By expressing $z = re^{j\omega} = e^{\sigma} e^{j\Omega}$ for $T_s = 1$, the radius is given by $r = e^{\sigma}$ so that if $\sigma < 0$ the singularities are inside the unit circle, if $\sigma = 0$ they are on the unit circle, and if $\sigma > 0$ they are outside the unit circle.

10.2 (a) We have $s[n] = u[n] - u[-n]$, the Z-transform of $s_1[n] = u[n]$ is

$$S_1(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}}$$

provided that $|z^{-1}| < 1$, thus $|z| > 1$ is the region of convergence for $S_1(z)$. The Z-transform of $s_2[n] = -u[-n]$ is given by

$$S_2(z) = -\sum_{n=-\infty}^{0} z^{-n} = -\sum_{m=0}^{\infty} z^{m} = \frac{-1}{1 - z}$$

where the last sum converges for $|z| < 1$.

(b) The condition for

$$S(z) = S_1(z) + S_2(z) = \frac{1}{1 - z^{-1}} - \frac{1}{1 - z} = \frac{1 + z^{-1}}{1 - z^{-1}}$$

to converge is that $|z| > 1$ and that $|z| < 1$ simultaneously, which is not possible. Since there is no region of convergence for $S(z)$, the Z-transform of $s[n]$ does not exist.

10.3 (a) The signal $x[n]$ can be written as

$$\begin{aligned} x[n] &= -\delta[n] - \alpha^{-1}\delta[n+1] - \alpha^{-2}\delta[n+2] - \cdots \\ &= -\delta[n] - \frac{1}{\alpha}\delta[n+1] - \frac{1}{\alpha^2}\delta[n+2] - \cdots \end{aligned}$$

So that its Z-transform is given by

$$\begin{aligned} X(z) &= -1 - \frac{z}{\alpha} - \frac{z^2}{\alpha^2} - \cdots \\ &= -\sum_{n=0}^{\infty}\left(\frac{z}{\alpha}\right)^n = -\frac{1}{1-z/\alpha} = \frac{\alpha z^{-1}}{1-\alpha z^{-1}} \qquad |z| < |\alpha| \end{aligned}$$

If $\alpha = 0.5$, the ROC is the interior of a circle of radius 0.5, which does not include the unit circle. The ROC in this case indicates that the signal is non-causal. If $\alpha = 2$, the ROC is the interior of a circle of radius 2, including the unit circle, and indicating the signal is non-causal. In this case $X(e^{j\omega})$ is defined, but it is not when $\alpha = 0.5$.

(b) Computing the derivative of $X(z)$ with respect to z gives

$$\begin{aligned} \frac{d\,X(z)}{dz} &= -\frac{1}{\alpha} - \frac{2z}{\alpha^2} - \frac{3z^2}{\alpha^3} \cdots = -\sum_{n=1}^{\infty} n\frac{z^{n-1}}{\alpha^n} \\ &= \sum_{m=-\infty}^{-1} m\alpha^m z^{-(m+1)} \end{aligned}$$

by letting $m = -n$ in the last sum. Letting now $k = m+1$ in the final sum we have

$$\frac{d\,X(z)}{dz} = \sum_{k=-\infty}^{0} (k-1)\alpha^{(k-1)} z^{-k}$$

We thus have the pair

$$\frac{d\,X(z)}{dz} \quad \Leftrightarrow \quad (n-1)\alpha^{(n-1)}u[-n].$$

Writing $X(z)$ in positive powers of z, i.e., $X(z) = \alpha/(z-\alpha)$ its derivative with respect to z is

$$\frac{d\,X(z)}{dz} = \frac{-\alpha}{(z-\alpha)^2}$$

so that we have

$$\frac{-\alpha}{(z-\alpha)^2} = \frac{-\alpha z^{-2}}{(1-\alpha z^{-1})^2} \quad \Leftrightarrow \quad (n-1)\alpha^{(n-1)}u[-n].$$

10.5 (a)(b) The given signal can also be written

$$x[n] = \begin{cases} 1 & n \geq 0 \text{ and even} \\ 0 & \text{otherwise} \end{cases}$$

Using the above expression for $x[n]$, we have

$$X(z) = \sum_{n=0,\text{ even}}^{\infty} 1\, z^{-n} = \sum_{m=0}^{\infty} 1\, z^{-2m} = \frac{1}{1 - z^{-2}} \qquad |z| > 1$$

where we let $n = 2m$ to find the final expression.
The z-transform of $x[n]$ is also obtained by using its linearity

$$\begin{aligned} X(z) &= 0.5\mathcal{Z}[u[n]] + 0.5\mathcal{Z}[(-1)^n u[n]] \\ &= \frac{1}{2(1 - z^{-1})} + 0.5 \sum_{n=0}^{\infty} (-z^{-1})^n \\ &= \frac{1}{2(1 - z^{-1})} + \frac{1}{2(1 + z^{-1})} \\ &= \frac{1}{1 - z^{-2}} \qquad |z^{-1}| < 1 \text{ or } |z| > 1 \end{aligned}$$

(c) To find the poles and zeros expressing $X(z)$ in positive powers:

$$X(z) = \frac{z^2}{z^2 - 1}$$

the poles are $z = \pm 1$, and the zeros $z = 0$, double.

10.7 (a) Writing $X(z)$ as

$$X(z) = \frac{1}{1-z^{-1}} - \frac{z^{-10}}{1-z^{-1}}$$

since the inverse Z-transform of the first term is $u[n]$, then the inverse of the second is $-u[n-10]$ given that z^{-10} indicates a delay of 10 samples. Thus,

$$x[n] = u[n] - u[n-10] = \begin{cases} 1 & 0 \leq n \leq 9 \\ 0 & \text{otherwise} \end{cases}$$

(b) Although $X(z)$ has been shown as a ratio of two polynomials, using the above representation of $x[n]$ its Z-transform is

$$X(z) = 1 + z^{-1} + \cdots + z^{-9}$$

i.e., a 9^{th}–order polynomial in z^{-1}.

(c) We can rewrite $X(z)$ as

$$X(z) = \frac{z^{10}-1}{z^9(z-1)} = \frac{(z-1)\prod_{k=1}^{9}(z-e^{j\pi k/5})}{z^9(z-1)} = \frac{\prod_{k=1}^{9}(z-e^{j\pi k/5})}{z^9}$$

which is obtained by finding that the zeros of $X(z)$ are values $z_k^{10} = 1$ or $z_k = e^{j2\pi k/10}$ for $k = 0, \cdots, 9$. For $k = 0$ the zero is $z_0 = 1$, which cancels the pole at 1.

10.10 (a) Pole is $z = 0.5$ and zero $z = -1$. ROC is (iii), $|z| > 0.5$ because of causality.

(b) From $H(z) = Y(z)/X(z)$ we have $Y(z) - 0.5Y(z)z^{-1} = X(z) + X(z)z^{-1}$ the difference equation is

$$y[n] - 0.5y[n-1] = x[n] + x[n-1]$$

(c) We can write

$$H(z) = \frac{1}{1 - 0.5z^{-1}} + \frac{z^{-1}}{1 - 0.5z^{-1}}$$

so that

$$\begin{aligned} h[n] &= 0.5^n u[n] + 0.5^{n-1}u[n-1] = \delta[n] + (0.5^n + 0.5^{n-1})u[n-1] \\ &= \delta[n] + 3 \times 0.5^n u[n-1] \end{aligned}$$

(d) The system is a low-pass filter. Drawing vectors from the pole and the zero to a point in the unit circle gives

$$H(e^{j\omega_0}) = \frac{\vec{Z}(e^{j\omega_0})}{\vec{P}(e^{j\omega_0})}$$

which gives $H(e^{j0}) = 2/0.5 = 4$, $H(e^{j\pi}) = 0$ and decreases as ω increases form 0 to π.

10.13 For $a = b = 2$ the denominator of the transfer function is

$$D(z) = z^2 + 2z + 2 = (z + 1)^2 + 1$$

so the poles are $p_{1,2} = -1 \pm j1$ which have $|p_{1,2}| = \sqrt{2} > 1$, outside the unit circle and as such system is unstable.

For $a = 1$ and $b = 1/2$ the denominator is

$$D(z) = z^2 + z + 0.5 = (z + 0.5)^2 + 0.25$$

so the poles are $p_{1,2} = -0.5 \pm j0.5$ which have $|p_{1,2}| = \sqrt{2}/2 < 1$, inside the unit circle and as such system is stable.

10.14 Impulse response

$$\begin{aligned} y[n] &= e[n-1] = x[n-1] - y[n-1] \\ H(z) &= \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1+z^{-1}} \\ h[n] &= (-1)^{n-1}u[n-1] \end{aligned}$$

System is unstable, $h[n]$ is not absolutely summable:

$$\sum_{n=1}^{\infty} \underbrace{|h[n]|}_{1} \to \infty$$

Moreover, the pole $z = 1$ is on the unit disc and as such the system is not BIBO stable.

10.15 (a) We have

$$\begin{aligned} Y(z) &= 1+2z^{-1}+2z^{-2}+z^{-3} \\ X(z) &= 1+z^{-1} \\ H(z) &= \frac{Y(z)}{X(z)} = 1+z^{-1}+z^{-2} \end{aligned}$$

after division.

(b) i. Yes, because of the ROC.

ii. There is a pole-zero cancellation

$$H(z) = \frac{1-0.5z^{-1}}{(1-0.5z^{-1})(1+0.5z^{-1})}$$

so that $h[n] = (-0.5)^n u[n]$.

(c) By long division in negative powers of z

$$H(z) = z^{-1} + 2z^{-2} + 2z^{-3} + \cdots + 2z^{-10000} + \cdots$$

so that $h[0] = 0$, $h[1] = 1$ and $h[10000] = 2$.
Another way is

$$H(z) = \frac{z^{-1}}{1-z^{-1}} + \frac{z^{-2}}{1-z^{-1}}$$

so that $h[n] = u[n-1] + u[n-2]$ giving the same values as above.

10.16 (a) The output can be written

$$y[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 3(-1)^n u[n-3]$$

where the last term can be expressed as $3(-1)^n u[n-3] = -3(-1)^{n-3} u[n-3]$ so that we can find its Z-transform by using the shift property, thus:

$$\begin{aligned} X(z) &= \frac{1}{1+z^{-1}} \\ Y(z) &= 1 + 2z^{-1} + 3z^{-2} - 3\frac{z^{-3}}{1+z^{-1}} \\ H(z) &= \frac{Y(z)}{X(z)} = (1+2z^{-1}+3z^{-2})(1+z^{-1}) - 3z^{-3} = 1 + 3z^{-1} + 5z^{-2} \end{aligned}$$

so $h[0] = 1$, $h[1] = 3$, $h[2] = 5$ and the others are zero.

(b) i. The transfer function is

$$H(z) = K\frac{(z+1)^2}{z}$$

where at dc $H(e^{j0}) = H(1) = 4K = 2$, so $K = 2/4 = 0.5$. As an FIR its ROC is the whole Z–plane except for $z = 0$.

ii. $x[n] = [\cos(0n) + \cos(\pi n/2) + \cos(\pi n)]u[n]$ the steady state response is

$$y_{ss}[n] = H(e^{j0}) + |H(e^{j\pi/2})|\cos(\pi n/2 + \angle H(e^{j\pi/2})) + |H(e^{j\pi})|\cos(\pi n + \angle H(e^{j\pi}))$$

where

$$\begin{aligned} &H(e^{j0}) = 2, \\ &H(e^{j\pi/2}) = 0.5\frac{(j+1)^2}{j} = 1e^{j0}, \\ &H(e^{j\pi}) = 0 \end{aligned}$$

and the steady–state is then

$$y_{ss}[n] = 2 + \cos(\pi n/2).$$

(c) i. The zero is $z = 0$ and the poles are $z = 0.5$ and $z = -2$, if $h[n]$ is non-causal then the region of convergence is ROC:$1/2 < |z| < 2$.

ii. If $h[n]$ is causal, then the ROC is $|z| > 2$ and

$$H(z) = \frac{z^{-1}}{(1-0.5z^{-1})(1+2z^{-1})} = \frac{A}{1-0.5z^{-1}} + \frac{B}{1+2z^{-1}}$$

where

$$\begin{aligned} A &= \frac{z^{-1}}{1+2z^{-1}}|_{z^{-1}=2} = \frac{2}{5} \\ B &= \frac{z^{-1}}{1-0.5z^{-1}}|_{z^{-1}=-1/2} = \frac{-2}{5} \end{aligned}$$

so the impulse response is

$$h[n] = (2/5)(0.5^n - (-2)^n)u[n]$$

iii. The ROC of convergence of the non-causal transfer function includes the unit circle, making the system BIBO stable, that is not the case for the causal transfer function and as it can be seen it is not BIBO stable.

10.18 (a) The output is

$$\begin{array}{ll} y[0]=0 & y[1]=1 \\ y[2]=3 & y[3]=4 \\ y[4]=4 & y[n]=3 \;\; n \geq 5 \end{array}$$

(b) $x[n]$ can be written as

$$\begin{aligned} x[n] &= n(u[n]-u[n-3])+u[n-3]) = \delta[n-1]+2\delta[n-2]+u[n-3] \\ X(z) &= z^{-1}+2z^{-2}+\frac{z^{-3}}{1-z^{-1}} \\ H(z) &= 1+z^{-1}+z^{-2} \end{aligned}$$

thus

$$Y(z) = X(z)H(z) = X(z)+z^{-1}X(z)+z^{-2}X(z)$$

so that

$$\begin{aligned} y[n] &= x[n]+x[n-1]+x[n-2] \\ &= \delta[n-1]+3\delta[n-2]+4\delta[n-3]+4\delta[n-4]+3u[n-5] \end{aligned}$$

10.24 (a) From the block diagram

$$Y(z) = X(z) + 0.5z^{-1}Y(z) + 0.25z^{-2}Y(z)$$

so that

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 0.5z^{-1} - 0.25z^{-2}}$$

(b) For $w[n] = x[n]$ then $R(z)H(z) = 1$ so that

$$R(z) = 1 - 0.5z^{-1} - 0.25z^{-2}$$

(c) $Y(z) = 1 + z^{-1}$, and

$$\begin{aligned} X(z) &= Y(z)R(z) \\ &= 1 + 0.5z^{-1} - 0.75z^{-2} - 0.25z^{-3} \end{aligned}$$

so that

$$x[n] = \delta[n] + 0.5\delta[n-1] - 0.75\delta[n-2] - 0.25\delta[n-3]$$