5.9 WHAT HAVE WE ACCOMPLISHED? WHAT IS NEXT?

By the end of this Chapter you should have a very good understanding of the frequency representation of signals and systems. In this Chapter, we have unified the treatment of periodic and non-periodic signals and their spectra and consolidated the concept of the frequency response of a linear time-invariant system. See Tables 5.1 and 5.2 for a summary of Fourier transform properties and pairs. Two significant applications are in filtering and in modulation. We introduced in this Chapter the fundamentals of modulation and its application in communications, as well as the basics of filtering. Both of these topics will be extended in Chapters 6 and 7.

Certainly the next step is to find out where the Laplace and the Fourier analysis apply. That will be done in the next two chapters. After that, we will go into discrete-time signals and systems. The concepts of sampling, quantization, and coding will bridge the continuous-time and the discrete-time and digital signals. Then transformations similar to Laplace and Fourier will permit us to do processing of discrete-time signals and systems.

Table 5.1 Basic Properties of Fourier Transform

	Time Domain	**Frequency Domain**
Signals and constants	$x(t), y(t), z(t), \alpha, \beta$	$X(\Omega), Y(\Omega), Z(\Omega)$
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(\Omega) + \beta Y(\Omega)$
Expansion/contraction in time	$x(\alpha t),\ \alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert} X\left(\frac{\Omega}{\alpha}\right)$
Reflection	$x(-t)$	$X(-\Omega)$
Parseval's energy relation	$E_x = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	$E_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(\Omega)\rvert^2 d\Omega$
Duality	$X(t)$	$2\pi x(-\Omega)$
Time differentiation	$\frac{d^n x(t)}{dt^n}, n \geq 1,$ integer	$(j\Omega)^n X(\Omega)$
Frequency differentiation	$-jtx(t)$	$\frac{dX(\Omega)}{d\Omega}$
Integration	$\int_{-\infty}^{t} x(t')dt'$	$\frac{X(\Omega)}{j\Omega} + \pi X(0)\delta(\Omega)$
Time shifting	$x(t - \alpha)$	$e^{-j\alpha\Omega} X(\Omega)$
Frequency shifting	$e^{j\Omega_0 t} x(t)$	$X(\Omega - \Omega_0)$
Modulation	$x(t) \cos(\Omega_c t)$	$0.5[X(\Omega - \Omega_c) + X(\Omega + \Omega_c)]$
Periodic signals	$x(t) = \sum_k X_k e^{jk\Omega_0 t}$	$X(\Omega) = \sum_k 2\pi X_k \delta(\Omega - k\Omega_0)$
Symmetry	$x(t)$ real	$\lvert X(\Omega)\rvert = \lvert X(-\Omega)\rvert$ $\angle X(\Omega) = -\angle X(-\Omega)$
Convolution in time	$z(t) = [x * y](t)$	$Z(\Omega) = X(\Omega)Y(\Omega)$
Windowing/Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}[X * Y](\Omega)$
Cosine transform	$x(t)$ even	$X(\Omega) = \int_{-\infty}^{\infty} x(t) \cos(\Omega t) dt,$ real
Sine transform	$x(t)$ odd	$X(\Omega) = -j \int_{-\infty}^{\infty} x(t) \sin(\Omega t) dt,$ imaginary

Table 5.2 Fourier Transform Pairs

	Function of Time	Function of Ω
(1)	$\delta(t)$	1
(2)	$\delta(t-\tau)$	$e^{-j\Omega\tau}$
(3)	$u(t)$	$\frac{1}{j\Omega}+\pi\delta(\Omega)$
(4)	$u(-t)$	$\frac{-1}{j\Omega}+\pi\delta(\Omega)$
(5)	$\text{sign}(t)=2[u(t)-0.5]$	$\frac{2}{j\Omega}$
(6)	$A,\ -\infty<t<\infty$	$2\pi A\delta(\Omega)$
(7)	$Ae^{-at}u(t),\ a>0$	$\frac{A}{j\Omega+a}$
(8)	$Ate^{-at}u(t),\ a>0$	$\frac{A}{(j\Omega+a)^2}$
(9)	$e^{-a\|t\|},\ a>0$	$\frac{2a}{a^2+\Omega^2}$
(10)	$\cos(\Omega_0 t),\ -\infty<t<\infty$	$\pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)]$
(11)	$\sin(\Omega_0 t),\ -\infty<t<\infty$	$-j\pi[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)]$
(12)	$p(t)=A[u(t+\tau)-u(t-\tau)],\ \tau>0$	$2A\tau\frac{\sin(\Omega\tau)}{\Omega\tau)}$
(13)	$\frac{\sin(\Omega_0 t)}{\pi t}$	$P(\Omega)=u(\Omega+\Omega_0)-u(\Omega-\Omega_0)$
(14)	$x(t)\cos(\Omega_0 t)$	$0.5[X(\Omega-\Omega_0)+X(\Omega+\Omega_0)]$

5.10 PROBLEMS

5.10.1 Basic Problems

5.1 A causal signal $x(t)$ having a Laplace transform with poles in the open-left s-plane (i.e., not including the $j\Omega$-axis) has a Fourier transform that can be found from its Laplace transform. Consider the following signals

$$x_1(t)=e^{-2t}u(t),\quad x_2(t)=r(t),\quad x_3(t)=x_1(t)x_2(t)$$

(a) Determine the Laplace transform of the above signals indicating their corresponding region of convergence.

(b) Determine for which of these signals you can find its Fourier transform from its Laplace transform. Explain.

(c) Give the Fourier transform of the signals that can be obtained from their Laplace transform.

Answers: (a) $X_2(s)=1/s^2$, $\sigma>0$; (b) $x_1(t)$ and $x_3(t)$.

5.2 There are signals whose Fourier transforms cannot be found directly by either the integral definition or the Laplace transform. For instance, the sinc signal

$$x(t)=\frac{\sin(t)}{t}$$

is one of them.