Signals and Systems

Fourier Series Part 2

- Reflection and even and odd periodic signals
- Real-valued periodic signals
- Time and frequency shifting
- Sum and multiplication of periodic signals
- Derivatives and integrals of periodic signals
- Amplitude and time scaling of periodic signals

- Book: Chapter 4
- Sections/subsections: 4.3.4, 4.3.6, 4.5

- Exercises: 4.2, 4.3, 4.4, 4.5, 4.7, 4.11 (3rd Ed.)
- Exercises: 4.2, 4.4, 4.6, 4.7, 4.10, 4.18 (2nd Ed.)

Reflection and even and odd periodic signals

- **Reflection:**
- Let $x(t)$ be a periodic signal with a fundamental period T_0 and a Fourier expansion

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}$$

- What is the Fourier expansion of $x(-t)$?

$$x(-t) = \sum_{k=-\infty}^{\infty} X_k e^{-jk\Omega_0 t} = \sum_{k=-\infty}^{\infty} X_{-k} e^{jk\Omega_0 t}$$

- Conclusion: if the Fourier coefficients of $x(t)$ are given by X_k then the Fourier coefficients of $x(-t)$ are given by X_{-k}

Reflection and even and odd periodic signals

- **Even periodic signals:**
- An even signal $x(t)$ is characterized by

$$x(-t) = x(t) \quad \text{for all } t \in \mathbb{R}$$

- Using the result of the previous slide we find that for an even signal

$$X_{-k} = X_k \quad \text{(even signal)}$$

- For the expansion coeffcients of the trigonometric Fourier series we have

$$c_k = \frac{X_k + X_{-k}}{2} = X_k \quad \text{and} \quad d_k = \mathrm{j}\frac{X_k - X_{-k}}{2} = 0$$

- The trigonometric Fourier series of an even signal has cosine expansion functions (even functions) only

Reflection and even and odd periodic signals

- **Odd periodic signals:**
- An odd signal $x(t)$ is characterized by

$$x(-t) = -x(t) \quad \text{for all } t \in \mathbb{R}$$

- For an odd signal we have

$$X_{-k} = -X_k \quad \text{(odd signal)}$$

- The expansion coeffcients of the trigonometric Fourier series are

$$c_k = \frac{X_k + X_{-k}}{2} = 0 \quad \text{and} \quad d_k = \mathrm{j}\frac{X_k - X_{-k}}{2} = \mathrm{j}X_k$$

- The trigonometric Fourier series of an odd signal has sine expansion functions (odd functions) only

Real-valued periodic signals

- Let $x(t)$ be a periodic signal with a fundamental period T_0 and a Fourier expansion

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}$$

- Taking the complex conjugate, we find

$$x^*(t) = \sum_{k=-\infty}^{\infty} X_k^* e^{-jk\Omega_0 t} = \sum_{k=-\infty}^{\infty} X_{-k}^* e^{jk\Omega_0 t}$$

- Conclusion: if the Fourier coefficients of $x(t)$ are given by X_k then the Fourier coefficients of $x^*(t)$ are given by X_{-k}^*

Real-valued periodic signals

- If the signal $x(t)$ is real-valued then $x^*(t) = x(t)$. Consequently, for a real-valued signal we have

$$X^*_{-k} = X_k \quad \text{or} \quad X_{-k} = X^*_k \quad \text{(real-valued signal)}$$

- If the signal $x(t)$ is *even and real-valued*, we have

$$X_k = X_{-k} = X^*_k$$

showing that the Fourier coefficients X_k are real. Note that the coefficients c_k are real as well.

Real-valued periodic signals

- If the signal $x(t)$ is *odd and real-valued*, we have

$$X_k = -X_{-k} = -X_k^*$$

  showing that the Fourier coefficients X_k are imaginary. Note that the coefficients d_k are real in this case.

Time and frequency shifting

- Let $x(t)$ be a periodic signal with a fundamental period T_0 and a Fourier expansion

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}$$

- What are the Fourier coefficients of $x(t-\tau)$, where $\tau \in \mathbb{R}$ is a time shift?

Time and frequency shifting

- From the Fourier expansion

$$x(t-\tau) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0(t-\tau)} = \sum_{k=-\infty}^{\infty} X_k e^{-jk\Omega_0\tau} e^{jk\Omega_0 t}$$

- Conclusion: if the Fourier coefficients of $x(t)$ are given by X_k then the Fourier coefficients of $x(t-\tau)$ are given by $X_k e^{-jk\Omega_0\tau}$.

Time and frequency shifting

- We are given a periodic signal $x(t)$ with fundamental period T_0. We consider the *modulated signal*

$$y(t) = x(t)e^{j\Omega_1 t}$$

- The frequency Ω_1 is called the *modulation frequency*
- For this frequency we take: $\Omega_1 = M\Omega_0$, with M an integer, $M \gg 1$
- The signal $y(t)$ is periodic with a fundamental period T_0

Time and frequency shifting

- For the signals $x(t)$ and $y(t)$ we have the Fourier expansions

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t} \quad \text{and} \quad y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{jk\Omega_0 t}$$

- How are the Fourier coefficients of $y(t)$ related to the Fourier coeffcients of $x(t)$?

$$y(t) = x(t)e^{j\Omega_1 t} = \sum_{k=-\infty}^{\infty} X_k e^{j(k\Omega_0 + \Omega_1)t}$$

$$= \sum_{k=-\infty}^{\infty} X_k e^{j(k+M)\Omega_0 t} = \sum_{k=-\infty}^{\infty} X_{k-M} e^{jk\Omega_0 t}$$

Time and frequency shifting

- Conclusion: if the Fourier coefficients of $x(t)$ are given by X_k then the Fourier coefficients of $x(t)e^{jM\Omega_0 t}$ are given by X_{k-M}.
- The spectrum of $x(t)$ is shifted in frequency by $\Omega_1 = M\Omega_0$ rad/s

Sum and multiplication of periodic signals

- Let $x(t)$ be a periodic signal with fundamental period T_1. Its fundamental frequency is

$$\Omega_1 = \frac{2\pi}{T_1}$$

- Let $y(t)$ be a periodic signal with fundamental period T_2. Its fundamental frequency is

$$\Omega_2 = \frac{2\pi}{T_2}$$

- Consider the signal $z(t)$, which is a linear combination of $x(t)$ and $y(t)$:

$$z(t) = \alpha x(t) + \beta y(t)$$

α and β are constants.

Sum and multiplication of periodic signals

- As we have seen, if

$$\frac{T_2}{T_1} = \frac{N}{M}$$

  with N and M integers ≥ 1 with no common factor then
- $z(t)$ is periodic with fundamental period and frequency

$$T_0 = MT_2 = NT_1, \quad \text{and} \quad \Omega_0 = \frac{2\pi}{T_0}$$

  respectively
- Note that

$$\Omega_1 = \frac{2\pi}{T_1} = N\Omega_0 \quad \text{and} \quad \Omega_2 = \frac{2\pi}{T_2} = M\Omega_0$$

Sum and multiplication of periodic signals

- Remark: when N and M have no common factor, then N and M are said to be *relatively prime* or *coprime*
- Example: $N = 4$ and $M = 6$ are *not* relatively prime. Common factor is 2.

$$\frac{4}{6} = \frac{2 \cdot 2}{2 \cdot 3} = \frac{2}{3}$$

- Example: $N = 2$ and $N = 3$ are relatively prime. These numbers do not have a common factor.

Sum and multiplication of periodic signals

- Since $x(t)$ is periodic with fundamental frequency $\Omega_1 = N\Omega_0$ it has a Fourier expansion of the form

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_1 t} = \sum_{k=-\infty}^{\infty} X_k e^{jkN\Omega_0 t}$$

- Since $y(t)$ is periodic with fundamental frequency $\Omega_2 = M\Omega_0$ it has a Fourier expansion of the form

$$y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{jk\Omega_2 t} = \sum_{k=-\infty}^{\infty} Y_k e^{jkM\Omega_0 t}$$

Sum and multiplication of periodic signals

- Since $z(t)$ is periodic with fundamental frequency Ω_0 it has a Fourier expansion of the form

$$z(t) = \sum_{k=-\infty}^{\infty} Z_k e^{jk\Omega_0 t}$$

- How are the coeffcients Z_k related to the coeffcients X_k and Y_k?

Sum and multiplication of periodic signals

- Using the Fourier expansions of $x(t)$ and $y(t)$, we find

$$\begin{aligned} z(t) &= \alpha x(t) + \beta y(t) \\ &= \sum_{k=-\infty}^{\infty} \alpha X_k e^{jkN\Omega_0 t} + \sum_{k=-\infty}^{\infty} \beta Y_k e^{jkM\Omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} Z_k e^{jk\Omega_0 t} \end{aligned}$$

- The integers: $\mathbb{Z}$
- Given an integer $N \geq 1$
- We say that an integer $k \in \mathbb{Z}$ is an integer multiple of N if there exists an integer $p \in \mathbb{Z}$ such that $k = pN$

Sum and multiplication of periodic signals

- Example: $N = 3$
  - $k = 0$ is an integer multiple of N, since for $p = 0 \in \mathbb{Z}$, we have $k = 0 \cdot N = 0$.
  - $k = 0$ is an integer multiple of any N
  - $k = 1$ is not an integer multiple of 3
  - $k = -1$ is not an integer multiple of 3
  - $k = 3$ is an integer multiple of $N = 3$ ($p = 1$)
  - $k = -3$ is an integer multiple of $N = 3$ ($p = -1$)

Sum and multiplication of periodic signals

- If k is an integer multiple of N **and** k is an integer multiple of M:
$$Z_k = \alpha X_{k/N} + \beta Y_{k/M}$$
- If k is *not* an integer multiple of N **and** k is *not* an integer multiple of M:
$$Z_k = 0$$
- If k is an integer multiple of N **and** k is *not* an integer multiple of M:
$$Z_k = \alpha X_{k/N}$$
- If k is *not* an integer multiple of N **and** k is an integer multiple of M:
$$Z_k = \beta Y_{k/M}$$

- Example: $N = 2$ and $M = 3$, $\alpha = \beta = 1$

$$Z_0 = X_0 + Y_0$$
$$Z_1 = 0$$
$$Z_2 = X_1$$
$$Z_3 = Y_1$$
$$Z_4 = X_2$$
$$Z_5 = 0$$
$$Z_6 = X_3 + Y_2$$
$$\ldots$$

Sum and multiplication of periodic signals

- Book:

$$z(t) = \sum_{k=-\infty}^{\infty} (\alpha X_{k/N} + \beta Y_{k/M}) e^{jk\Omega_0 t}$$

with k/N and k/M integers

Sum and multiplication of periodic signals

- Let $x(t)$ and $y(t)$ be periodic signals with fundamental period T_0
- Fourier expansions of these signals

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t} \quad \text{and} \quad y(t) = \sum_{m=-\infty}^{\infty} Y_m e^{jm\Omega_0 t}$$

- We multiply the signals $x(t)$ and $y(t)$ to obtain

$$z(t) = x(t)y(t)$$

- The signal $z(t)$ is also periodic with fundamental period T_0

Sum and multiplication of periodic signals

- Fourier expansion of $z(t)$:

$$z(t) = \sum_{n=-\infty}^{\infty} Z_n e^{jn\Omega_0 t}$$

- How are the Fourier coefficients of $x(t)$ and $y(t)$ related to the Fourier coefficients of $z(t)$?

Sum and multiplication of periodic signals

- We compute

$$
\begin{aligned}
z(t) &= x(t)y(t) \\
&= \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t} \sum_{m=-\infty}^{\infty} Y_m e^{jm\Omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} X_k Y_m e^{j(k+m)\Omega_0 t} \\
&\overset{n=k+m}{=} \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} X_k Y_{n-k} e^{jn\Omega_0 t} \\
&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} X_k Y_{n-k} e^{jn\Omega_0 t} = \sum_{n=-\infty}^{\infty} Z_n e^{jn\Omega_0 t}
\end{aligned}
$$

Sum and multiplication of periodic signals

- We conclude that the Fourier coefficients of $z(t)$ are given by

$$Z_n = \sum_{k=-\infty}^{\infty} X_k Y_{n-k}$$

- Z_n is equal to the convolution of the discrete sequences X_k and Y_k
- Compare with

$$z(t) = \int_{\tau=-\infty}^{\infty} x(\tau) y(t-\tau)\, \mathrm{d}\tau$$

- Discrete convolutions will be discussed extensively further on in the course

Derivatives and integrals of periodic signals

- Periodic signal $x(t)$ with a Fourier expansion

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}$$

- Let $y(t)$ be the derivative of this signal. We have

$$y(t) = \frac{dx}{dt} = \sum_{k=-\infty}^{\infty} X_k \cdot jk\Omega_0 \cdot e^{jk\Omega_0 t} = \sum_{k=-\infty}^{\infty} Y_k e^{jk\Omega_0 t}$$

- We conclude

$$Y_k = X_k \cdot jk\Omega_0$$

Derivatives and integrals of periodic signals

- Periodic signal $y(t)$ with a Fourier expansion

$$y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{jk\Omega_0 t}$$

- Signal has no dc component: $Y_0 = 0$

$$y(t) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} Y_k e^{jk\Omega_0 t}$$

Derivatives and integrals of periodic signals

- Integral of $y(t)$:

$$z(t) = \int_{\tau=-\infty}^{t} y(\tau)\, \mathrm{d}\tau$$

- With $MT_0 \leq t$ and M an integer, we have

$$\begin{aligned} z(t) &= \int_{\tau=-\infty}^{t} y(\tau)\, \mathrm{d}\tau \\ &= \underbrace{\int_{\tau=-\infty}^{MT_0} y(\tau)\, \mathrm{d}\tau}_{=0} + \int_{\tau=MT_0}^{t} y(\tau)\, \mathrm{d}\tau \\ &= \int_{\tau=MT_0}^{t} y(\tau)\, \mathrm{d}\tau \end{aligned}$$

Derivatives and integrals of periodic signals

- Substitute the Fourier series of $y(t)$ to obtain

$$
\begin{aligned}
z(t) &= \int_{\tau=MT_0}^{t} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} Y_k e^{jk\Omega_0\tau}\, d\tau \\
&= \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} Y_k \int_{\tau=MT_0}^{t} e^{jk\Omega_0\tau}\, d\tau \\
&= -\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} Y_k \frac{1}{jk\Omega_0} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} Y_k \frac{1}{jk\Omega_0} e^{jk\Omega_0 t} \\
&= Z_0 + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} Z_k e^{jk\Omega_0 t}
\end{aligned}
$$

Derivatives and integrals of periodic signals

- with

$$Z_0 = -\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} Y_k \frac{1}{jk\Omega_0}$$

and

$$Z_k = Y_k \frac{1}{jk\Omega_0} \quad k \neq 0$$

Amplitude and time scaling of periodic signals

- Periodic signal $x(t)$ with a Fourier expansion

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}$$

- What is the Fourier transform of $y(t) = Ax(\alpha t)$, $\alpha > 0$?
- From the Fourier expansion of $x(t)$:

$$y(t) = Ax(\alpha t) = \sum_{k=-\infty}^{\infty} AX_k e^{jk\alpha\Omega_0 t}$$

- We observe:
  - $y(t)$ is a periodic signal with fundamental frequency $\alpha\Omega_0$
  - and Fourier coefficients $Y_k = AX_k$
  - Note that time scaling with an $\alpha > 0$ does not affect the Fourier coeffcients