# Chapter 4 – Frequency Analysis: The Fourier Series

**Richard Heusdens**

# Fourier series

Jean Baptiste Joseph Fourier (1768-1830)

Fourier's idea:

*any* periodic function can be written as a weighted sum of sines and cosines of different frequencies.

(not exactly true)

# Fourier series

$$x(t) = c_0 + 2\sum_{k=1}^{\infty}\left(c_k\cos(k\Omega_0 t) + d_k\sin(k\Omega_0 t)\right), \qquad \Omega_0 = \frac{2\pi}{\underbrace{T_0}_{\text{period}}}$$

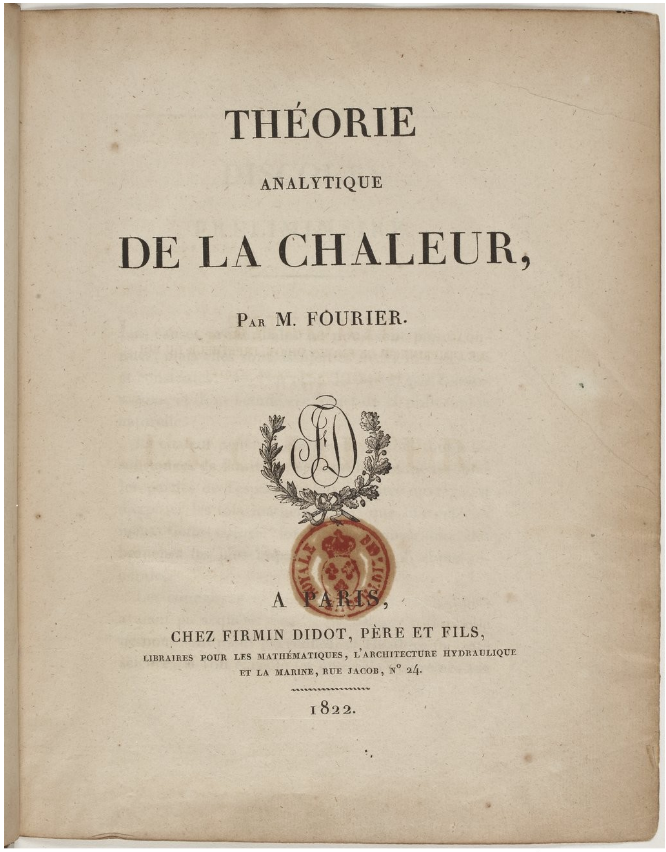
When Fourier submitted his paper in 1807, the committee (which included Lagrange, Laplace, Malus and Legendre, among others) concluded:

... *the manner in which the author arrives at these equations is not exempt of difficulties and [...] his analysis to integrate them still leaves something to be desired on the score of generality and even rigour.*

# Fourier series

Fourier's paper never got published, until some 15 years later, when Fourier wrote his own book, *The Analytical Theory of Heat* (Fourier 1822).

In that book, Fourier extended his finding to non-periodic signals, stating that such a signal can be represented by a weighted integral of a series of sine and cosine functions. Such an integral is termed the *Fourier transform*.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ANALYTIQUE
DE LA CHALEUR,
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A PARIS,
CHEZ FIRMIN DIDOT, PÈRE ET FILS,
LIBRAIRES POUR LES MATHÉMATIQUES, L'ARCHITECTURE HYDRAULIQUE ET LA MARINE, RUE JACOB, N° 24.
1822.

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# What in this chapter?

- Eigenfunctions and LTI systems
- Complex and trigonometric Fourier series
- Spectrum of periodic signals
- Fourier series and Laplace transform
- Properties of Fourier series
- Convergence of Fourier series

# Eigenfunctions revisited

Consider a LTI system with input signal $x(t) = e^{s_0 t}, t \in \mathbb{R}$:

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{s_0(t-\tau)} d\tau$$

$$= e^{s_0 t} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-s_0 \tau} d\tau}_{H(s_0)} = e^{s_0 t} H(s_0)$$

transfer function

assuming $H(s_0)$ exists ($s_0 \in$ ROC)

- $e^{s_0 t}$ is called an *eigenfunction* of the LTI system

# Eigenfunctions revisited

Special case:

- $e^{s_0 t} = e^{j\Omega_0 t}$: harmonic signal $\quad \Rightarrow \quad y(t) = H(j\Omega_0)e^{j\Omega_0 t}$

The function $H(j\Omega)$ is called the *frequency response* of the LTI system:

$$y(t) = H(j\Omega_0)e^{j\Omega_0 t} = |H(j\Omega_0)|e^{j(\Omega_0 t + \angle H(j\Omega_0))}$$

- *magnitude response* $|H(j\Omega_0)|$ modifies the magnitude of $e^{j\Omega_0 t}$
- *phase response* $\angle H(j\Omega_0)$ modifies the phase of $e^{j\Omega_0 t}$

# Fourier series and transform

Express $x$ as a linear combination of harmonics $e^{j\Omega_k t}$:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j\Omega_k t} \quad \Rightarrow \quad t) = \sum_{k=-\infty}^{\infty} H(j\Omega_k) X_k e^{j\Omega_k t}$$

Fourier series

or

Fourier transform

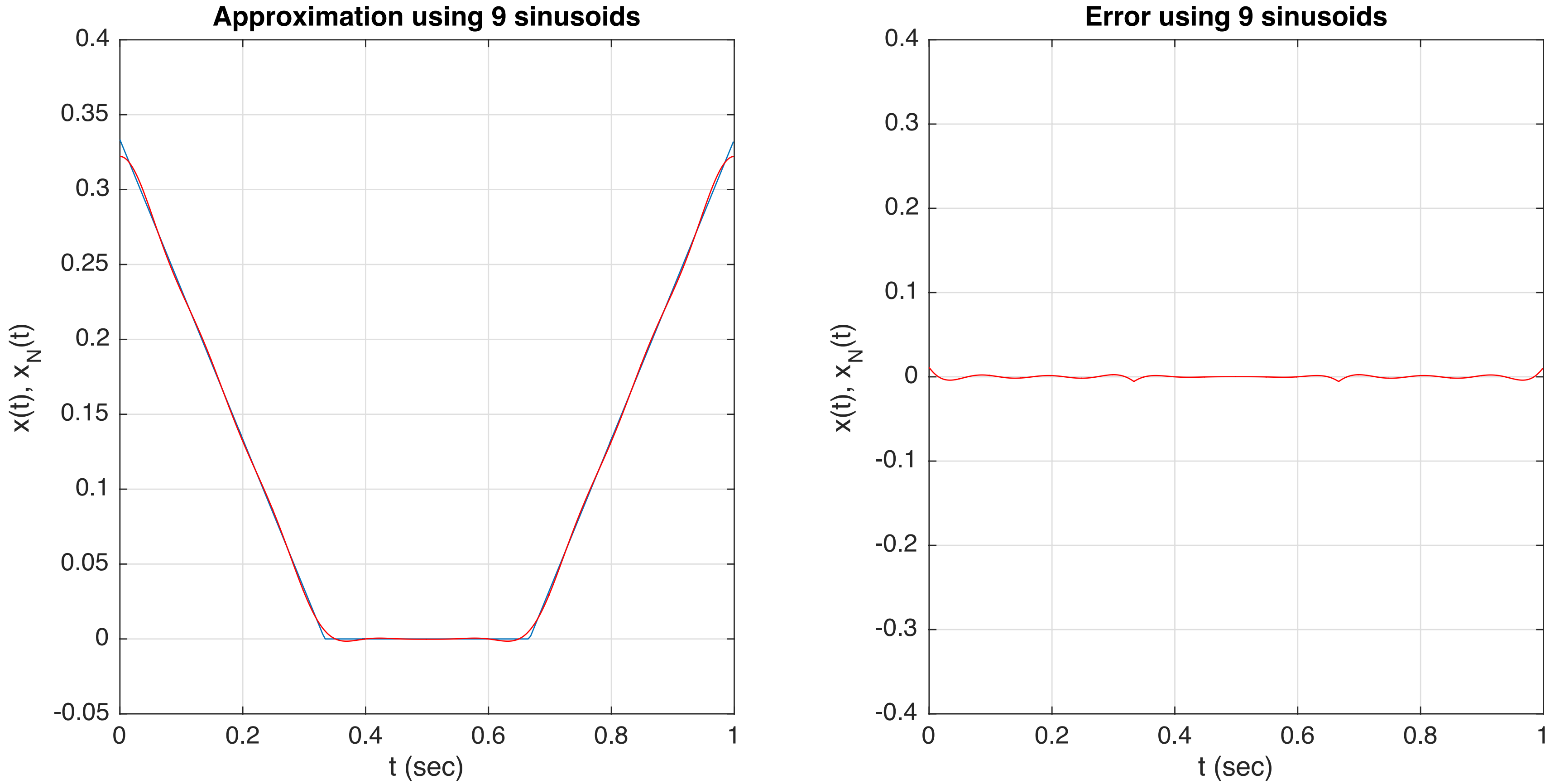
$$x(t) = \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega \quad \Rightarrow \quad y(t) = \int_{-\infty}^{\infty} H(j\Omega) X(\Omega) e^{j\Omega t} d\Omega$$

# Why study this special case?

- Fourier transform is probably the most widely applied signal processing tool in science and engineering
- Ties together two of the most used phenomenas known to engineers: those of time and frequency
- Time and frequency are dual domains
- Many signal manipulations are done in the frequency domain including filtering, sampling, modulation, etc.
- Harmonic signals appear naturally in many applications
- Steady-state analysis

# Fourier series

Can we represent/approximate any signal by sinusoids?



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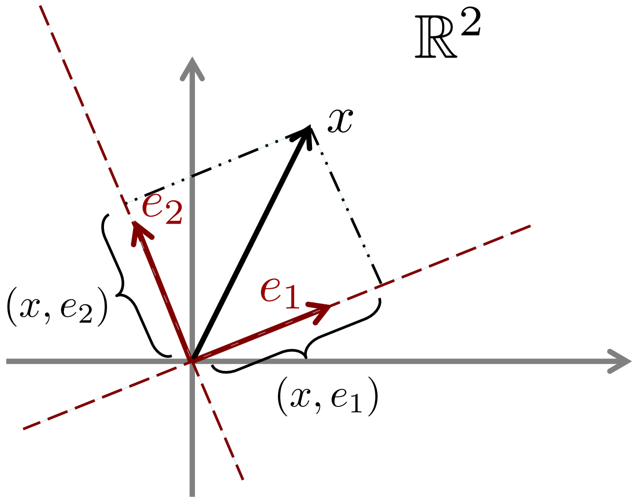
# Fourier series

Key questions:

- Does *any* signal has such a representation?
- How should we choose the frequencies of the constituent sinusoids?
- How do we find the weights?
- How many do we need, finite or infinite many?
- How does the sequence converge (in norm, pointwise, uniformly)?

# Orthonormal system




$$(e_k, e_m) = \begin{cases} 1, & k = m \\ 0, & k \neq m \end{cases}$$

Pythagoras' theorem

$$x = \sum_{k=1}^{2} (x, e_k) e_k, \qquad \|x\|^2 = \sum_{k=1}^{2} |(x, e_k)|^2$$

# Orthonormal system

- The functions $\{\psi_k(t), t \in [a,b]\}$ are called *orthonormal* (orthogonal and normalized) if

$$(\psi_k, \psi_m) = \int_a^b \psi_k(t)\psi_m^*(t)dt = \begin{cases} 0 & k \neq m \\ 1 & k = m \end{cases}$$

- To ensure the existence of the norm and the inner product, we assume the functions have finite energy. That is, $\psi_k \in L^2([a,b])$ where

$$L^2(E) = \{f : \int_E |f(t)|^2 dt < \infty\}$$

# Orthonormal system

**Theorem:** Let $\{\psi_k\}_{k=1}^{\infty}$ denote a complete orthonormal system in $L^2(E)$ and let $x \in L^2(E)$. Then

$$x = \sum_{k=1}^{\infty} (x, \psi_k)\psi_k$$

Moreover, we have (Parseval's identity)

$$\|x\|^2 = \sum_{k=1}^{\infty} |(x, \psi_k)|^2$$

# Complex Fourier series

The *Fourier series representation* of a periodic signal $x(t)$ of period $T_0$ is given by

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}, \quad \Omega_0 = \frac{2\pi}{T_0}$$

with *Fourier coefficients* $X_k$

$$X_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jk\Omega_0 t} dt, \quad k \in \mathbb{Z}$$

Moreover, we have *(Parseval's identity)*: $\|x\|^2 = T_0 \sum_{k=-\infty}^{\infty} |X_k|^2$

# Complex Fourier series

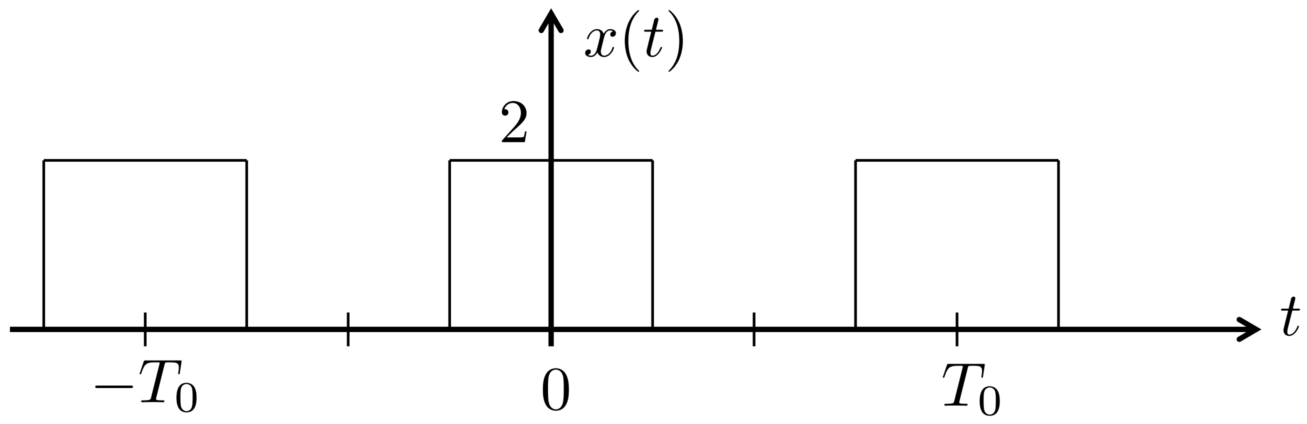
- Fourier series determine the frequency components of periodic signals and how the power is distributed over the frequency components, called the *spectrum*
- The spectrum of periodic signals is *discrete* (line spectrum)
- For real signals, the spectrum is conjugate symmetric:

$$X_{-k} = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{jk\Omega_0 t} dt = X_k^*$$

- As a consequence, $\underbrace{|X_{-k}| = |X_k|}_{\text{even symmetric}}$ and $\underbrace{\angle X_{-k} = -\angle X_k}_{\text{odd symmetric}}$

# Complex Fourier series

**Example** (periodic pulse train):




- DC (average) value of 1

# Complex Fourier series

- Complex Fourier series

$$X_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jk\Omega_0 t}dt = \frac{2}{T_0}\int_{-T_0/4}^{T_0/4} e^{-jk\Omega_0 t}dt$$

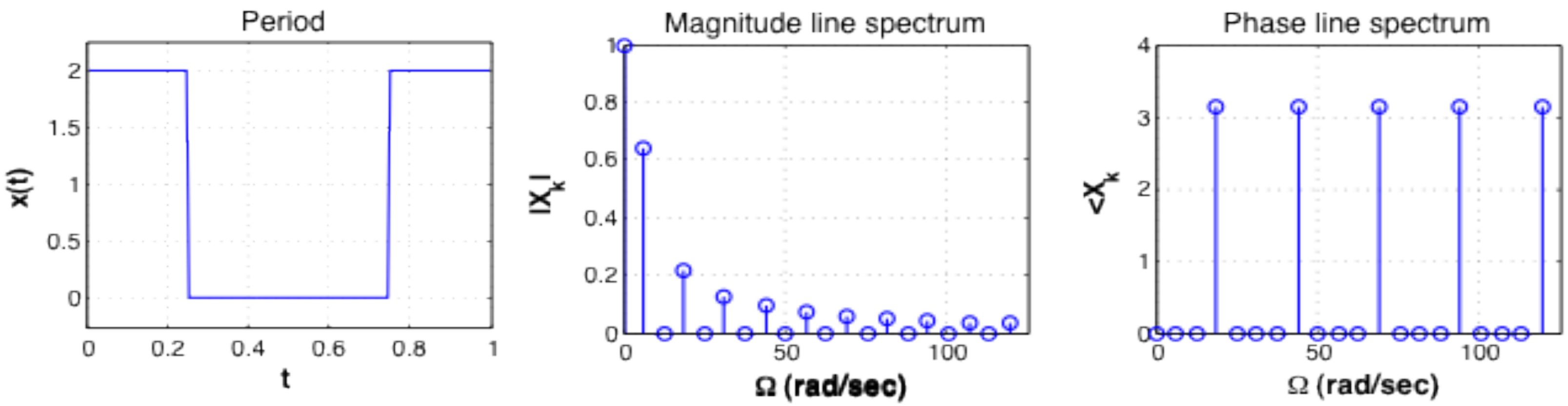
$$= \begin{cases} \dfrac{-1}{jk\pi}\, e^{-jk\Omega_0 t}\Big|_{-T_0/4}^{T_0/4}, & k \neq 0 \\ 1, & k = 0 \end{cases}$$

$$= \begin{cases} \dfrac{2}{k\pi}\sin(k\pi/2), & k \neq 0 \\ 1, & k = 0 \end{cases}$$

# Complex Fourier series

Fourier series is given by

$$x(t) = 1 + \frac{2}{\pi}e^{j\Omega_0 t} + \frac{2}{\pi}e^{-j\Omega_0 t} - \frac{2}{3\pi}e^{j3\Omega_0 t} - \frac{2}{3\pi}e^{-j3\Omega_0 t} \dots$$




*Period of train of rectangular pulses and its magnitude and phase line spectra.*

# Complex Fourier series

Note that $|X_k| = |X_{-k}|$ and that $\angle X_k = -\angle X_{-k}$.

Hence, we can rewrite the Fourier series as

$$\begin{aligned} x(t) &= 1 + \frac{2}{\pi}e^{j\Omega_0 t} + \frac{2}{\pi}e^{-j\Omega_0 t} - \frac{2}{3\pi}e^{j3\Omega_0 t} - \frac{2}{3\pi}e^{-j3\Omega_0 t} \dots \\ &= 1 + \frac{4}{\pi}\cos(\Omega_0 t) - \frac{4}{3\pi}\cos(3\Omega_0 t) + \cdots \end{aligned}$$

# Trigonometric Fourier series

The *Fourier series representation* of a periodic signal $x(t)$ of period $T_0$ is given by

$$x(t) = c_0 + 2\sum_{k=0}^{\infty} \left(c_k \cos(k\Omega_0 t) + d_k \sin(k\Omega_0 t)\right), \quad \Omega_0 = \frac{2\pi}{T_0}$$

with *Fourier coefficients* $c_k$ and $d_k$

$$c_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos(k\Omega_0 t)dt, \quad k = 0, 1, 2, \cdots$$

$$d_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\sin(k\Omega_0 t)dt, \quad k = 1, 2, \cdots$$

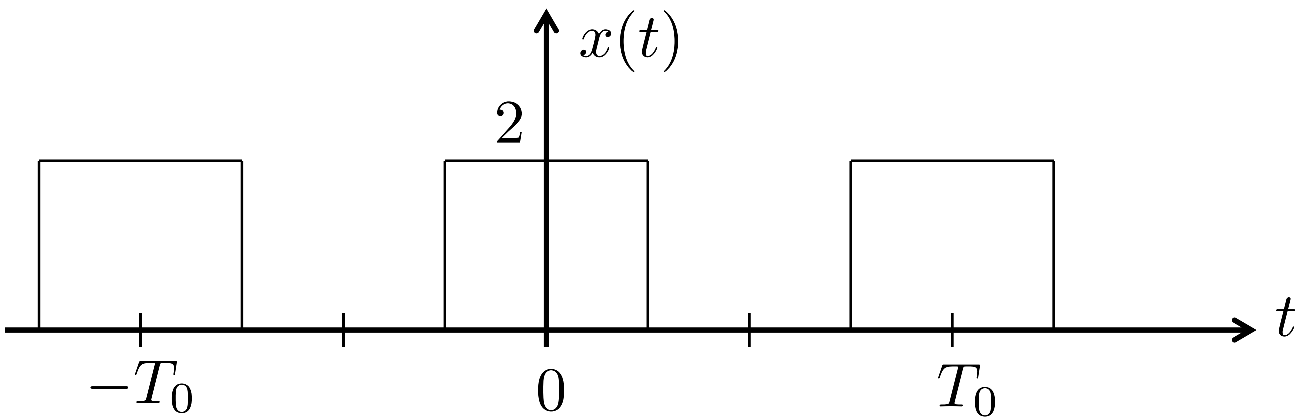
# Trigonometric Fourier series

Observe that

- $c_k = \frac{1}{2}(X_k + X_{-k}),\ d_k = \frac{j}{2}(X_k - X_{-k})$
- $X_k = c_k - jd_k,\ X_{-k} = c_k + jd_k$
- $|X_k| = \sqrt{c_k^2 + d_k^2},\ \angle X_k = -\tan^{-1}\left(\frac{d_k}{c_k}\right)$
- if $x$ even symmetric $(x(t) = x(-t))$, all $d_k$s are zero
- if $x$ odd symmetric $(x(t) = -x(-t))$, all $c_k$s are zero

# Trigonometric Fourier series

**Example** (periodic pulse train):




- DC (average) value of 1
- $x(t)$ is even symmetric ($d_k$s are zero)

# Trigonometric Fourier series

- Trigonometric Fourier series

$$c_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\cos(k\Omega_0 t)dt = \frac{4}{T_0}\int_0^{T_0/4}\cos(k\Omega_0 t)dt$$

$$= \begin{cases} \frac{2}{k\pi}\sin(k\pi/2), & k \neq 0 \\ 1, & k = 0 \end{cases}$$

Hence, the Fourier series is given by

$$x(t) = 1 + \frac{4}{\pi}\cos(\Omega_0 t) - \frac{4}{3\pi}\cos(3\Omega_0 t) + \cdots$$

# Fourier series

Some observations:

- Notice that

$$\lim_{k \to \infty} X_k = \lim_{k \to \infty} c_k = 0$$

**Riemann-Lebesgue lemma**: If $x \in L^2(E)$, then

$$\lim_{k \to \pm\infty} \int_E x(t) e^{-jk\Omega_0 t} dt = 0$$

$$\lim_{k \to \infty} \int_E x(t) \cos(k\Omega_0 t) dt = \lim_{k \to \infty} \int_E x(t) \sin(k\Omega_0 t) dt = 0$$

# Fourier series

Some observations:

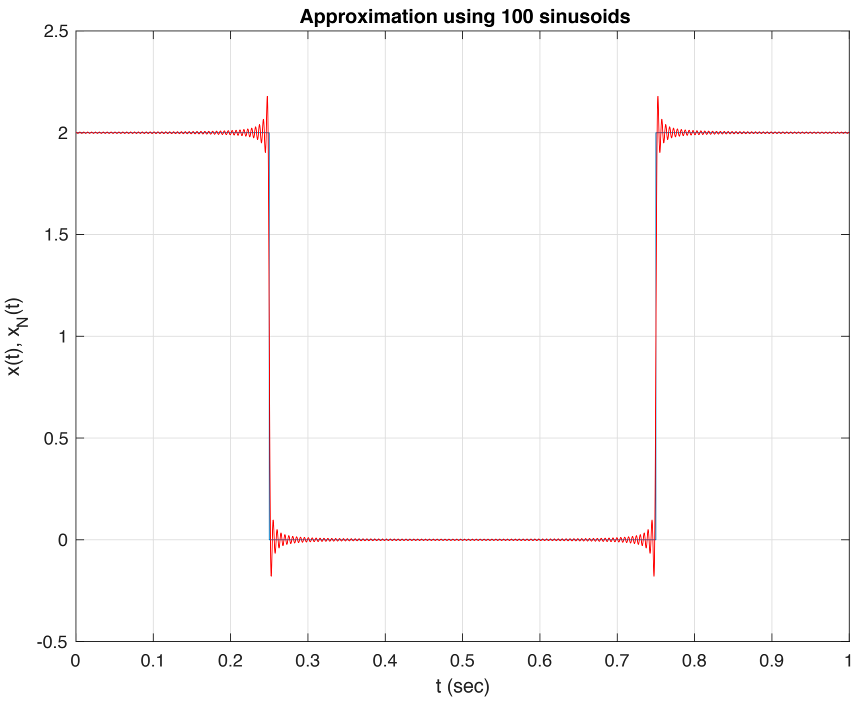
- Notice that
$$\lim_{k\to\infty} X_k = \lim_{k\to\infty} c_k = 0$$
and that the decay is of order $\mathcal{O}(k^{-1})$

- $x(T_0/4) = 2$ but the Fourier series yields 1 at $t = T_0/4$ ???

- The Fourier series seems to have convergence problems around discontinuities (*Gibb's phenomena*)

# Gibb's Phenomena

$$x(t) = 1 + \frac{4}{\pi}\cos(\Omega_0 t) - \frac{4}{3\pi}\cos(3\Omega_0 t) + \cdots$$



# Convergence of Fourier series

Some (important) remarks:

- Let $y(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}$ denotes the Fourier series representation of $x(t)$. Then $\|x - y\| = 0$. That is

$$\int_E |x(t) - y(t)|^2 dt = 0$$

- We have convergence *in norm*, which does not imply that it converges *pointwise* to $x(t)$

# Relation Laplace transform

Let $x_1(t)$ be defined as

$$x_1(t) = \begin{cases} x(t), & t \in [t_0, t_0 + T_0] \\ 0, & \text{otherwise} \end{cases}$$

one fundamental period

Then

$$X_1(s) = \int_{t_0}^{t_0+T_0} x_1(t) e^{-st} dt.$$

$$X_k = \frac{1}{T_0} X_1(s)\Big|_{s=jk\Omega_0}$$

The Fourier coefficients $X_k$ are given by

$$X_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x_1(t) e^{-jk\Omega_0 t} dt$$

# Properties of Fourier Series

- Time-shifting: $\quad y(t) = x(t+\tau) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad Y_k = e^{jk\Omega_0\tau} X_k$

  Direct approach:

$$Y_k = \frac{1}{T_0}\int_0^{T_0} x(t+\tau)e^{-jk\Omega_0 t}dt = \frac{1}{T_0}\int_0^{T_0} x(s)e^{-jk\Omega_0(s-\tau)}ds$$
$$= \frac{e^{jk\Omega_0\tau}}{T_0}\int_0^{T_0} x(s)e^{-jk\Omega_0 s}ds = e^{jk\Omega_0\tau}X_k$$

  Using Laplace (see Table 3.1):

$$Y_k = \frac{1}{T_0}Y_1(s)\bigg|_{s=jk\Omega_0} = \frac{1}{T_0}\left(e^{s\tau}X_1(s)\right)\bigg|_{s=jk\Omega_0} = e^{jk\Omega_0\tau}X_k$$

# Properties of Fourier Series

- Frequency-shifting: $y(t) = x(t)e^{-jm\Omega_0 t} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad Y_k = X_{k+m}$

Direct approach:

$$Y_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jm\Omega_0 t}e^{-jk\Omega_0 t}dt = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(k+m)\Omega_0 t}dt = X_{k+m}$$

Using Laplace (see Table 3.1):

$$Y_k = \frac{1}{T_0}Y_1(s)\Big|_{s=jk\Omega_0} = \frac{1}{T_0}X_1(s+jm\Omega_0)\Big|_{s=jk\Omega_0} = X_{k+m}$$

# Properties of Fourier Series

- Differentiation: $\quad y(t) = \dfrac{dx(t)}{dt} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad Y_k = jk\Omega_0 X_k$

Direct approach:

$$Y_k = \frac{1}{T_0}\int_0^{T_0} x'(t)e^{-jk\Omega_0 t}dt = \frac{1}{T_0}\int_0^{T_0} e^{-jk\Omega_0 t}d(x(t))$$

$$= \frac{1}{T_0}\, x(t)e^{-jk\Omega_0 t}\Big|_0^{T_0} + \frac{jk\Omega_0}{T_0}\int_0^{T_0} x(t)e^{-jk\Omega_0 t}dt = jk\Omega_0 X_k$$

Using Laplace (see Table 3.1):

$$Y_k = \frac{1}{T_0}Y_1(s)\Big|_{s=jk\Omega_0} = \frac{1}{T_0}\left(sX_1(s)\right)\Big|_{s=jk\Omega_0} = jk\Omega_0 X_k$$

# Convergence of the Fourier Series

- Fourier series can be defined for functions $x \in L^1(E) \supset L^2(E)$
- The pointwise convergence of a Fourier series is a rather complicated problem

  (Dirichlet conditions)

  – Dirichlet (1829) showed that if $x \in L^1(E)$ and has a finite number of discontinuities and extrema, then the Fourier series converges everywhere to the local average

  – Kolmogorov (1926) has given an example of a function in $L^1(E)$ in which the Fourier series diverges everywhere!

  – Carleson (1966) proved that if $x \in L^2(E)$, then the Fourier series converges for almost all $t$ to $x(t)$

# Convergence of the Fourier Series

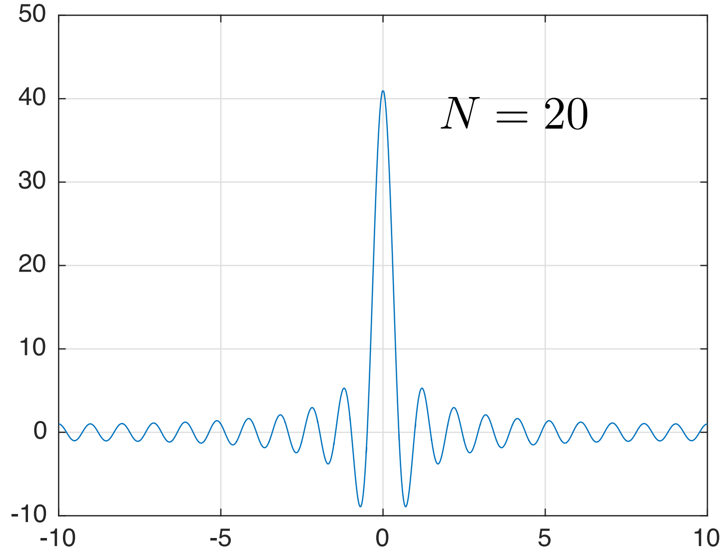
Dirichlet's theorem: if $x \in L^1([0, T_0])$, then

$$S_N(t) = \frac{1}{T_0} \int_0^{T_0} x(u) D_N(u-t) du$$

where

$$D_N(t) = \frac{\sin((N+\frac{1}{2})\Omega_0 t)}{\sin(\frac{1}{2}\Omega_0 t)}$$

is called *Dirichlet's kernel*



# Convergence of the Fourier Series

Conclusion:

- Convergence of Fourier series depends on *local* behaviour
- For large $N$, the Dirichlet kernel becomes a $\delta$-function:

$$\lim_{N\to\infty} S_N(t_0) = \frac{1}{2}\left(x(t_0^-) + x(t_0^+)\right)$$

If $x$ is continuous at $t = t_0$, then $\lim_{N\to\infty} S_N(t_0) = x(t_0)$

In conclusion, the Fourier series converges in *norm* (we have equality in $L^2([0, T_0])$), but we only have *pointwise convergence* at points where $x$ is continuous!

# Convergence of the Fourier Series

$\Omega_0 = 1$

**Example:** $x(t) = t, t \in [-\pi, \pi]$. Since $x$ is odd, $c_k = 0$ for all $k$ and

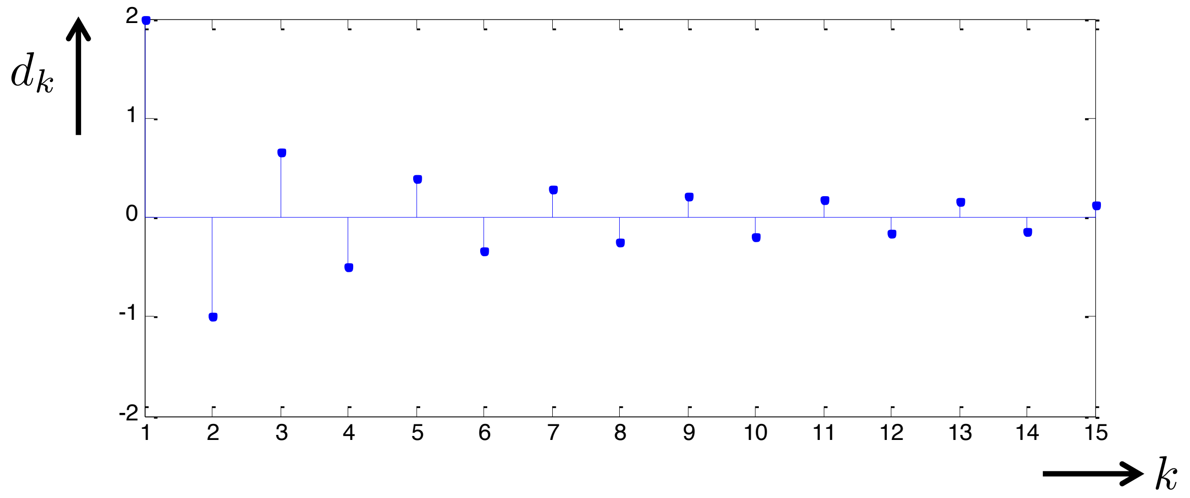
$$
\begin{aligned}
d_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} t\sin(kt)dt \\
&= \frac{-1}{2\pi k} t\cos(kt)\Big|_{-\pi}^{\pi} + \frac{1}{2\pi k}\underbrace{\int_{-\pi}^{\pi}\cos(kt)dt}_{=0} = \frac{1}{k}(-1)^{k+1}
\end{aligned}
$$

Hence, the Fourier series becomes

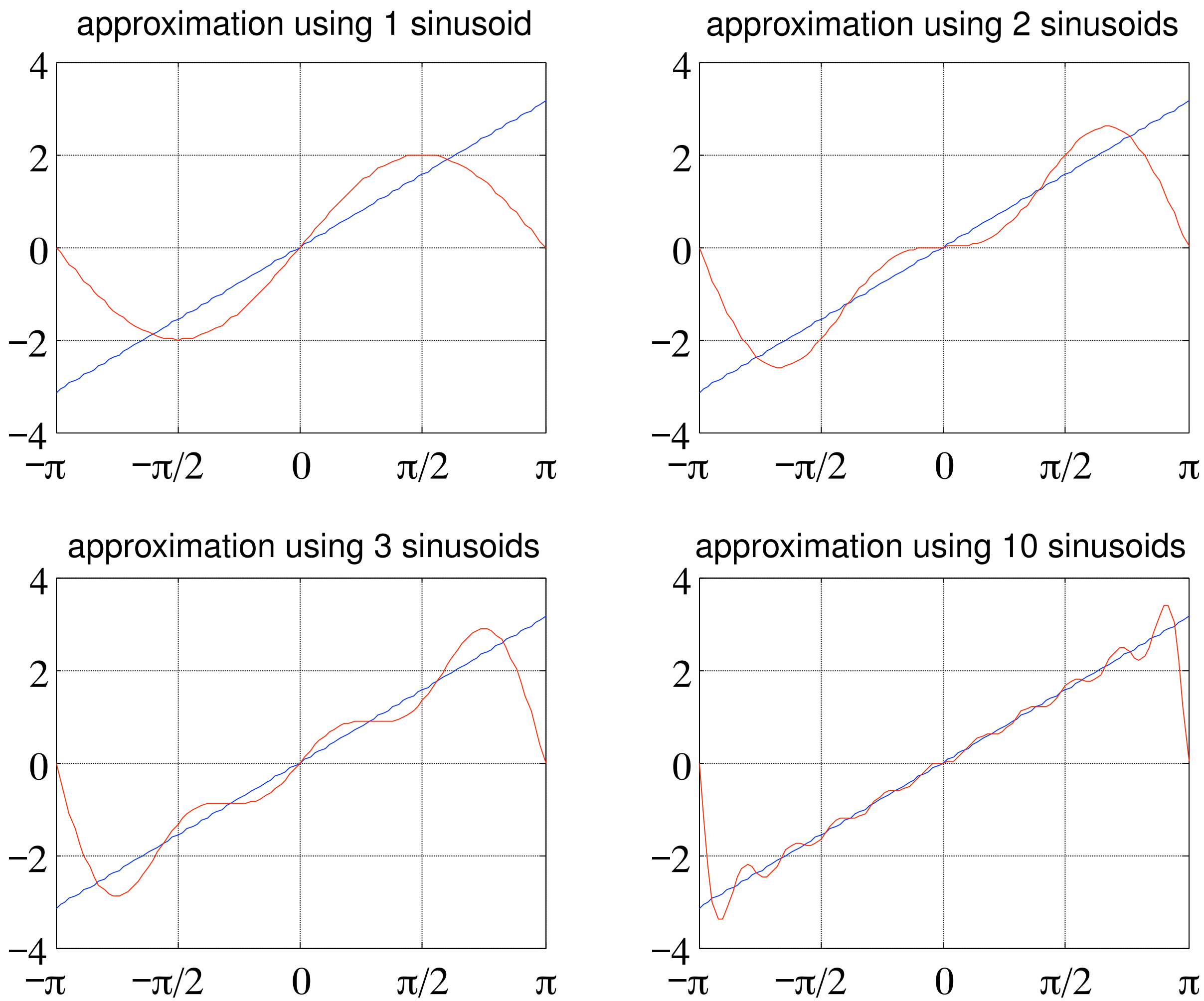
$$
\frac{2}{1}\sin(t) - \frac{2}{2}\sin(2t) + \frac{2}{3}\sin(3t) - \frac{2}{4}\sin(4t) + \cdots
$$

# Convergence of the Fourier Series

- Notice that $\lim_{k \to \infty} d_k = 0$ and that the decay is $\mathcal{O}(k^{-1})$
- $x(\pi) = \pi$ but substituting $t = \pi$ in the Fourier series yields $0$



# Convergence of the Fourier Series



# Convergence of the Fourier Series

**Example:** $x(t) = |t|, t \in [-\pi, \pi]$. Since $x$ is even, $d_k = 0$ for all $k$

$$c_k = \frac{1}{\pi}\int_0^\pi t\cos(kt)dt$$

$$\overset{(k\neq 0)}{=} \underbrace{\frac{1}{k\pi} t\sin(kt)\Big|_0^\pi}_{=0} - \frac{1}{k\pi}\int_0^\pi \sin(kt)dt = \begin{cases} 0, & k = 2, 4, \ldots \\ \frac{-2}{\pi k^2}, & k \text{ odd} \end{cases}$$

and

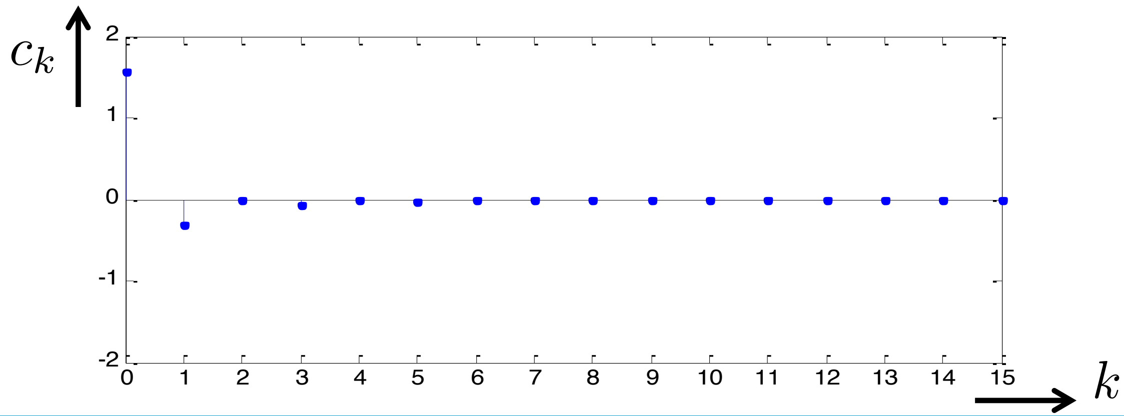
$$c_0 = \frac{1}{\pi}\int_0^\pi t dt = \frac{\pi}{2}$$

# Convergence of the Fourier Series

The Fourier series becomes

$$\frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos(t)}{1^2} + \frac{\cos(3t)}{3^2} + \frac{\cos(5t)}{5^2} + \cdots\right)$$

Notice that $\lim_{k\to\infty} c_k = 0$ and that the decay is of order $\mathcal{O}(k^{-2})$



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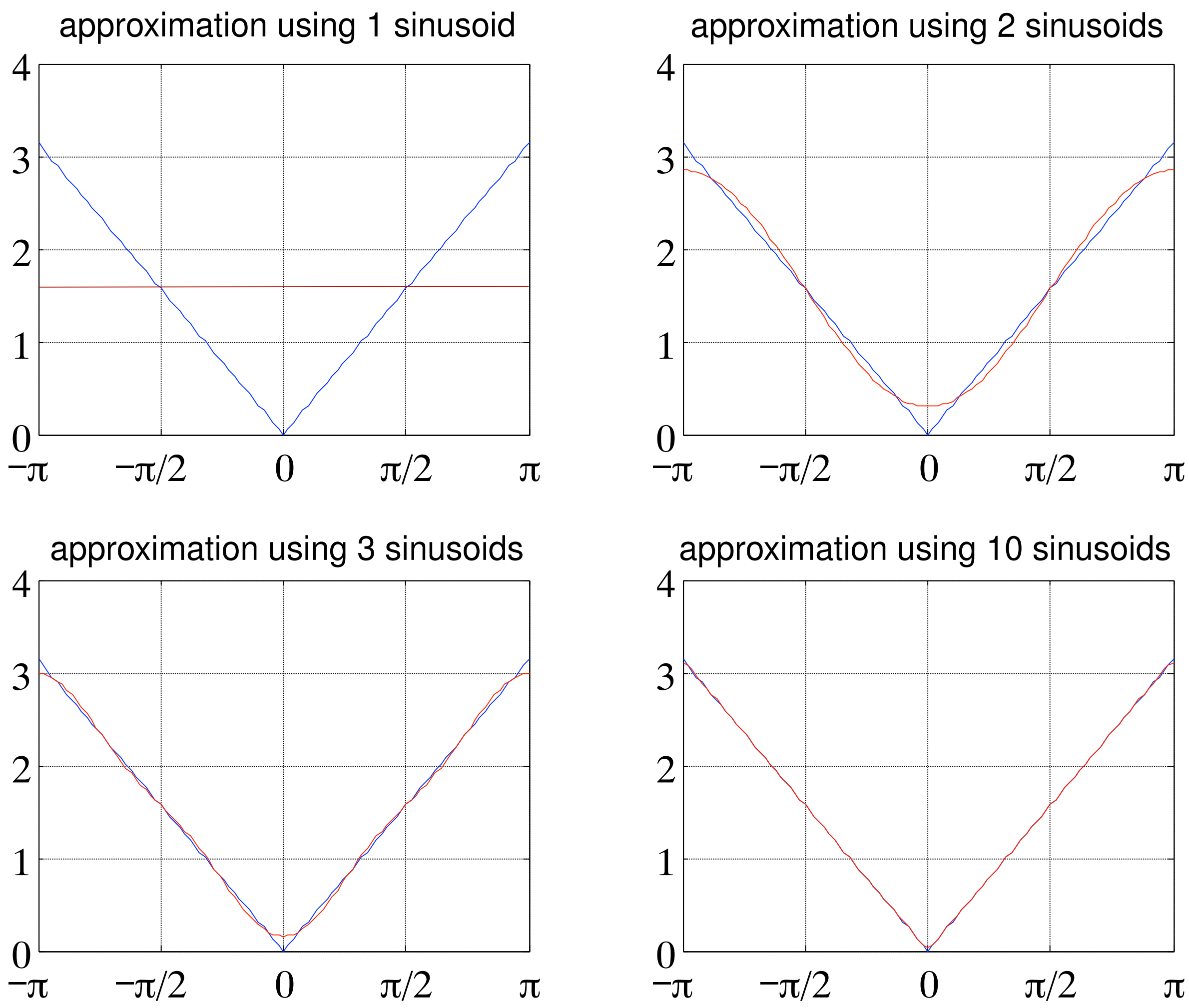
# Convergence of the Fourier Series



# Convergence of the Fourier Series

- If $x$ is $p$ times differentiable and all derivatives are in $L^1(E)$, then

$$x^{(p)}(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad (jk\Omega_0)^p X_k$$

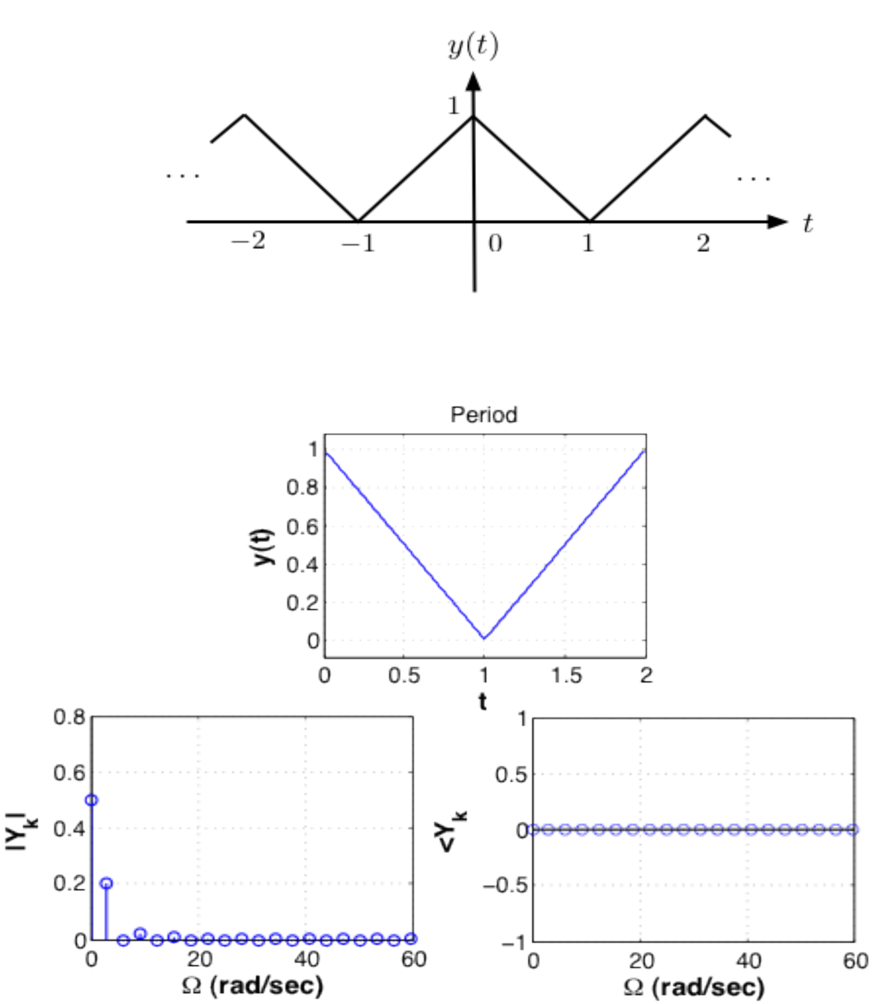
- Applying the Riemann-Lebesgue lemma on $x^{(p)}$, we conclude that

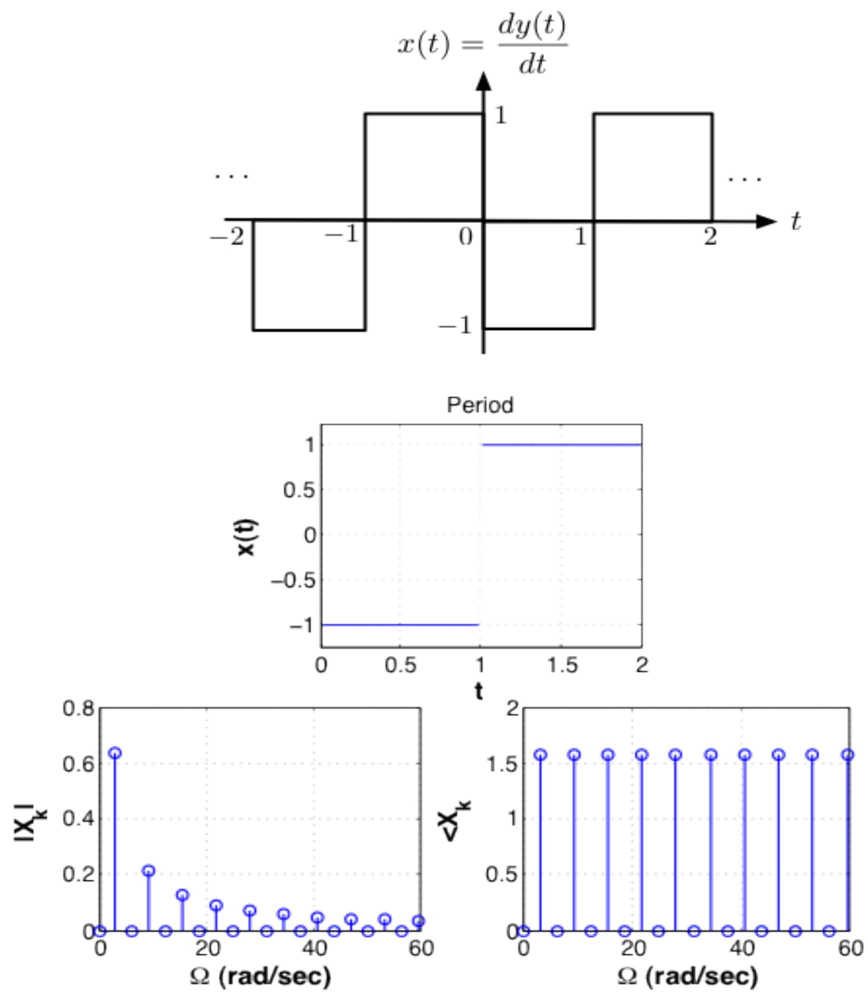
$$\lim_{k\to\pm\infty} (k\Omega_0)^p X_k = 0$$

  so that regularity of $x$ translates to rapid decay of $X_k$

This explains the faster decay of the Fourier coefficients of the function $x(t) = |t|$ as compared to those of $x(t) = t$.

# Convergence of the Fourier Series



# What have we accomplished?

- Response of LTI systems to periodic signals (eigenfunction property)
- Harmonic (sinusoidal) representation of periodic/finite-length signals
- Spectrum of periodic/finite-length signals
- Connection between Fourier and Laplace
- Convergence properties of Fourier series

# Where do we go?

- Extension of Fourier representation for aperiodic/infinite-length signals
- Unification of spectral theory for periodic and aperiodic signals
- Connection between Fourier and Laplace transforms
- Duality relation time and frequency domain
- Convolution and filtering
- Relation between pole/zero locations and frequency response