Signals and Systems

Fourier Series Part 1

Introducting Fourier series

- Introduction to Fourier series
- The complex exponential Fourier series
- Convergence of the Fourier series
- Parseval's power relation
- Trigonometric Fourier series
- Fourier series and the Laplace transform
- Response of LTI systems to periodic signals

Introduction to Fourier series

- Book: Chapter 4
- Sections/subsections: 4.1, 4.2, 4.3.1, 4.3.2, 4.3.3, 4.3.5, 4.4

- Exercises: 4.2, 4.3, 4.4, 4.5, 4.7, 4.11 (3rd Ed.)
- Exercises: 4.2, 4.4, 4.6, 4.7, 4.10, 4.18 (2nd Ed.)

Introduction to Fourier series

- We have seen that the exponential signal is an eigensignal of an LTI system
- We now focus on periodic signals and use this exponential signal to describe such functions
- Recall that a signal $x(t)$ is periodic if there exists a $T > 0$ such that
$$x(t+T) = x(t) \quad \text{for all } t \in \mathbb{R}$$
- T is called a period of the signal
- The smallest period is denoted as T_0 and is called the *fundamental period*

Introduction to Fourier series

- We start by constructing periodic signals using exponential signals as building blocks
- Let us start with the signal

$$x_1(t) = X_1 e^{j\Omega_0 t} + X_{-1} e^{-j\Omega_0 t}$$

  - X_1 and X_{-1} are complex numbers
  - Ω_0 [rad/s] is the *fundamental frequency* of the signal
  - The signal has a fundamental period

$$T_0 = \frac{2\pi}{\Omega_0}$$

Introduction to Fourier series

- We provide the numbers X_1 and X_{-1} to realize the signal $x_1(t)$
- **Example:** $X_1 = X_{-1} = 1/2$:

$$x_1(t) = \cos(\Omega_0 t)$$

- **Example:** $X_1 = X_{-1}^* = \frac{1}{2j}$:

$$x_1(t) = \sin(\Omega_0 t)$$

- What if we add a constant?

$$x_1(t) = X_0 + X_1 e^{j\Omega_0 t} + X_{-1} e^{-j\Omega_0 t}$$

- Signal is still periodic with fundamental period T_0
- What if we add additional powers of the exponential signal?

$$x_N(t) = \sum_{k=-N}^{N} X_k e^{jk\Omega_0 t}$$

- Signal is still periodic with fundamental period T_0

- Note the procedure up till now: We provide the X_k's to construct $x_N(t)$
- Now the other way around
- Suppose
  - we know $x_N(t)$
  - and we know that $x_N(t)$ can be written in the form

$$x_N(t) = \sum_{k=-N}^{N} X_k e^{jk\Omega_0 t}$$

- We do not know the coefficients X_k, however

- How do we determine these coefficients?
- **Step 1:** Start with

$$x_N(t) = \sum_{k=-N}^{N} X_k e^{jk\Omega_0 t}$$

- **Step 2:** Multiply this equation by $e^{-jm\Omega_0 t}$, m an integer, $|m| \leq N$

$$e^{-jm\Omega_0 t} x_N(t) = \sum_{k=-N}^{N} X_k e^{j(k-m)\Omega_0 t}$$

- Integrate over a single period:

$$\int_{t=t_0}^{t_0+T_0} e^{-\mathrm{j}m\Omega_0 t} x_N(t)\,\mathrm{d}t = \int_{t=t_0}^{t_0+T_0} \sum_{k=-N}^{N} X_k e^{\mathrm{j}(k-m)\Omega_0 t}\,\mathrm{d}t$$

$$= \sum_{k=-N}^{N} X_k \int_{t=t_0}^{t_0+T_0} e^{\mathrm{j}(k-m)\Omega_0 t}\,\mathrm{d}t$$

- Since

$$\int_{t=t_0}^{t_0+T_0} e^{\mathrm{j}(k-m)\Omega_0 t}\,\mathrm{d}t = \begin{cases} T_0 & m = k \\ 0 & m \neq k \end{cases}$$

- We are left

$$\int_{t=t_0}^{t_0+T_0} e^{-jm\Omega_0 t} x_N(t)\,dt = T_0 X_m$$

- and find

$$X_m = \frac{1}{T_0}\int_{t=t_0}^{t_0+T_0} x_N(t) e^{-jm\Omega_0 t}\,dt, \quad m = 0, \pm 1, \pm 2, ..., \pm N$$

- **Conclusion:**
- A periodic signal $x_N(t)$ is given and it is known that it can be written in the form

$$x_N(t) = \sum_{k=-N}^{N} X_k e^{jk\Omega_0 t} \qquad (*)$$

- The coefficients can be determined as

$$X_k = \frac{1}{T_0} \int_{t=t_0}^{t_0+T_0} x_N(t) e^{-jk\Omega_0 t}\, dt, \quad k = 0, \pm 1, \pm 2, \ldots, \pm N$$

- The signal of Eq. $(*)$ is known as a *finite Fourier series*

- Note that $x_N(t)$ is a very smooth function of time
- It can be differentiated arbitrarily often and the resulting signal is continuous again
- Now what if we have a periodic signal with a discontinuity?
- Or what if we have a periodic signal with a derivative that has a discontinuity?
- Or what if we have a periodic signal for which its nth derivative ($n \geq 1$) has a discontinuity?

The complex exponential Fourier series

- To make a chance of representing such signals by exponential signals, we take an *infinite* number of exponential expansion signals
- We write

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}$$

with

$$X_k = \frac{1}{T_0} \int_{t=t_0}^{t_0+T_0} x_N(t) e^{-jk\Omega_0 t}\, dt, \quad k = 0, \pm 1, \pm 2, \ldots$$

- This is the *complex exponential Fourier series* of the periodic signal $x(t)$

Convergence of the Fourier series

- Some remarks about convergence
- When discussing convergence of the Fourier series, the basic question to answer is:
  - What happens to the partial sums

$$x_N(t) = \sum_{k=-N}^{N} X_k e^{jk\Omega_0 t} \quad \text{as } N \to \infty?$$

Convergence of the Fourier series

- **Pointwise convergence:** Let $x(t)$ be a periodic signal with fundamental period T_0. The signal is piecewise continuous with a piecewise continuous derivative.
- If $x(t)$ is continuous at $t = t_0$, then

$$x(t_0) = \lim_{N \to \infty} x_N(t_0) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t_0}$$

- If $x(t)$ has a jump discontinuity at $t = t_0$ with left limit $x(t_0^-)$ and right limit $x(t_0^+)$, then

$$\frac{x(t_0^-) + x(t_0^+)}{2} = \lim_{N \to \infty} x_N(t_0) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t_0}$$

- Other convergence definitions
- **Uniform convergence:**

$$\max_{t_0 \leq t \leq t_0 + T_0} |x(t) - x_N(t)| \to 0 \quad \text{as } N \to \infty$$

Loosely speaking, when the signal $x_N(t)$ converges uniformly to $x(t)$, then the graph of $x_N(t)$ "stays close" to the graph of $x(t)$ on the complete interval $t_0 \leq t \leq t_0 + T_0$

- Convergence in the sense that the average quadratic error tends to zero as $N \to \infty$:

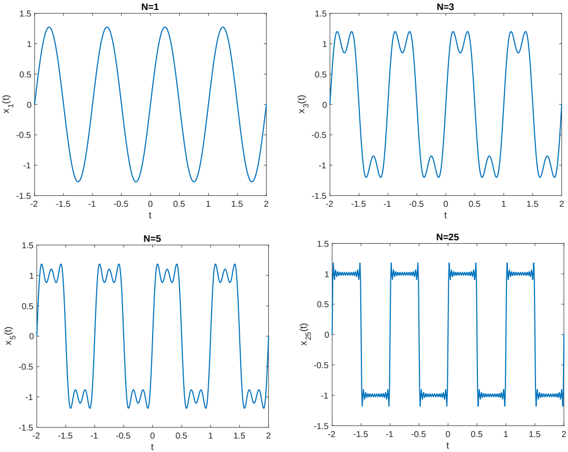
$$\lim_{N\to\infty} \frac{1}{T_0} \int_{t=t_0}^{t_0+T_0} |x(t) - x_N(t)|^2 \, dt = 0$$

- Type of convergence depends on the signal
- Uniform convergence is the strongest type of convergence. It implies pointwise and averaged squared error convergence

Convergence of the Fourier series

- **Gibb's phenomenon**

$$x(t) = \begin{cases} 1 & 0<t<1/2, \\ -1 & 1/2<t<1 \end{cases}$$



Parseval's power relation

- Recall that the power of a periodic signal $x(t)$ is given by

$$P_x = \frac{1}{T_0} \int_{t=t_0}^{t_0+T_0} |x(t)|^2 \, dt$$

- If $x(t)$ is square integrable then $P_x < \infty$
- For $x(t)$ we have the Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}$$

Parseval's power relation

- For its complex conjugate, we have

$$x^*(t) = \sum_{m=-\infty}^{\infty} X_m^* e^{-jm\Omega_0 t}$$

- Consequently,

$$\begin{aligned} |x(t)|^2 &= x(t)x^*(t) \\ &= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} X_k X_m^* e^{j(k-m)\Omega_0 t} \end{aligned}$$

Parseval's power relation

- Substitution gives

$$P_x = \frac{1}{T_0} \int_{t=t_0}^{t_0+T_0} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} X_k X_m^* e^{j(k-m)\Omega_0 t}\, dt$$

$$= \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} X_k X_m^* \int_{t=t_0}^{t_0+T_0} e^{j(k-m)\Omega_0 t}\, dt$$

Parseval's power relation

- Since

$$\int_{t=t_0}^{t_0+T_0} e^{\mathrm{j}(k-m)\Omega_0 t}\,\mathrm{d}t = \begin{cases} T_0 & m = k \\ 0 & m \neq k \end{cases}$$

- we arrive at

$$P_x = \sum_{k=-\infty}^{\infty} |X_k|^2$$

- This is *Parseval's power relation*

Parseval's power relation

- Parseval's power relation stated differently
- Write

$$x(t) = \sum_{k=-\infty}^{\infty} x_k(t) \quad \text{with} \quad x_k(t) = X_k e^{jk\Omega_0 t}$$

- We have

$$P_{x_k} = |X_k|^2$$

- In words: the power of the signal $x(t)$ is equal to the sum of powers of its Fourier series components

- **Power line spectrum:**

$$\text{Plot } |X_k|^2 \text{ vs. } k\Omega_0, \quad k = 0, \pm 1, \pm 2, \ldots \;.$$

- **Magnitude line spectrum:**

$$\text{Plot } |X_k| \text{ vs. } k\Omega_0, \quad k = 0, \pm 1, \pm 2, \ldots \;.$$

- **Phase line spectrum:**

$$\text{Plot } \angle X_k \text{ vs. } k\Omega_0, \quad k = 0, \pm 1, \pm 2, \ldots \;.$$

Parseval's power relation

- Consider a signal that is square integrable, that is, it has finite power
- Parseval's power relation

$$\sum_{k=-\infty}^{\infty} |X_k|^2 = P_x < \infty$$

- The sum on the left-hand side converges
- Consequently,

$$|X_k|^2 \to 0 \quad \text{as } k \pm \infty$$

- In words: the Fourier coefficients tend to zero as $k \to \pm\infty$

Parseval's power relation

- It can be shown that if the signal is absolutely integrable then
$$\lim_{k\to\infty} X_k = 0$$
as well. This is the famous *Riemann-Lebesgue lemma*
- Can we say something about how fast the coefficients tend to zero as $k \to \pm\infty$?

Parseval's power relation

- For simplicity, consider a signal $x(t)$
  - having a jump discontinuity at $t = \tilde{t}$, $t_0 < \tilde{t} < t_0 + T_0$
  - Left limit: $x(\tilde{t}^-)$, right limit: $x(\tilde{t}^+)$
  - No jumps at the end points: $x(t_0) = x(t_0 + T_0)$
  - Away from $\tilde{t}$, $x(t)$ has continuous derivatives up to any desired order

- For the Fourier coeffcients, we have

$$
\begin{aligned}
X_k &= \frac{1}{T_0}\int_{t=t_0}^{t_0+T_0} x(t)e^{-jk\Omega_0 t}\,dt \\
&= \frac{1}{T_0}\int_{t=t_0}^{\tilde{t}} x(t)e^{-jk\Omega_0 t}\,dt + \frac{1}{T_0}\int_{t=\tilde{t}}^{t_0+T_0} x(t)e^{-jk\Omega_0 t}\,dt
\end{aligned}
$$

- First integral. Integration by parts gives

$$\begin{aligned}\frac{1}{T_0}\int_{t=t_0}^{\tilde{t}} x(t)e^{-jk\Omega_0 t}\,dt &= \frac{1}{j2\pi k}e^{-jk\Omega_0 t_0}x(t_0) \\ &- \frac{1}{j2\pi k}e^{-jk\Omega_0 \tilde{t}^-}x(\tilde{t}^-) \\ &+ \frac{1}{j2\pi k}\int_{t=t_0}^{\tilde{t}} x'(t)e^{-jk\Omega_0 t}\,dt\end{aligned}$$

- where we have used $T_0\Omega_0 = 2\pi$

Parseval's power relation

- Second integral. Integration by parts gives

$$\begin{aligned}\frac{1}{T_0}\int_{t=\tilde{t}}^{t_0+T_0} x(t)e^{-jk\Omega_0 t}\,dt &= \frac{1}{j2\pi k}e^{-jk\Omega_0 \tilde{t}^+}x(\tilde{t}^+)\\ &-\frac{1}{j2\pi k}e^{-jk\Omega_0 t_0}x(t_0+T_0)\\ &+\frac{1}{j2\pi k}\int_{t=\tilde{t}}^{t_0+T_0} x'(t)e^{-jk\Omega_0 t}\,dt\end{aligned}$$

- where we have used $T_0\Omega_0 = 2\pi$

Parseval's power relation

- Consequently,

$$X_k = \frac{1}{j2\pi k} e^{-jk\Omega_0 t} x(t) \Big|_{\tilde{t}^-}^{\tilde{t}^+} + \frac{1}{j2\pi k} \int_{t=t_0}^{t_0+T_0} x'(t) e^{-jk\Omega_0 t} \, dt$$

- Since $x(t)$ has a jump discontinuity at $t = \tilde{t}$, the first term on the right-hand side does not vanish
- We conclude that the Fourier coeffcient X_k must at least have a $1/k$ term

Parseval's power relation

- Now what if $x(t)$ is continuous at $t = \tilde{t}$, but its derivative has a jump discontinuity at $t = \tilde{t}$?
- Since $x(t)$ is continuous at $t = \tilde{t}$, the first term on the right-hand side now vanishes
- In this case, we have

$$X_k = \frac{1}{\mathrm{j}2\pi k} \int_{t=t_0}^{t_0+T_0} x'(t) e^{-\mathrm{j}k\Omega_0 t}\, dt$$

- Follow a similar procedure as above (integrate by parts again)
- In this case, we find that the Fourier coeffcient X_k must at least have a $1/k^2$ term

Parseval's power relation

- **Summary:**

- $x(t)$ has a jump discontinuity at $t = \tilde{t}$:

$$X_k \text{ should at least have a } 1/k \text{ term}$$

- $x(t)$ is continuous, but $x'(t)$ has a jump discontinuity at $t = \tilde{t}$:

$$X_k \text{ should at least have a } 1/k^2 \text{ term}$$

- $x(t)$ and $x'(t)$ are continuous, but $x''(t)$ has a jump discontinuity at $t = \tilde{t}$:

$$X_k \text{ should at least have a } 1/k^3 \text{ term}$$

- and so on

Trigonometric Fourier series

- We rewrite the complex Fourier series expansion in terms of cos/sin expansion functions
- The analysis is straightforward

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t} \\
&= \sum_{k=-\infty}^{-1} X_k e^{jk\Omega_0 t} + X_0 + \sum_{k=1}^{\infty} X_k e^{jk\Omega_0 t} \\
&= X_0 + \sum_{k=1}^{\infty} X_{-k} e^{-jk\Omega_0 t} + \sum_{k=1}^{\infty} X_k e^{jk\Omega_0 t}
\end{aligned}
$$

Trigonometric Fourier series

- We now use Euler's formula to obtain

$$x(t) = X_0 + \sum_{k=1}^{\infty} X_{-k}[\cos(k\Omega_0 t) - \mathrm{j}\sin(k\Omega_0 t)] + \sum_{k=1}^{\infty} X_k[\cos(k\Omega_0 t) + \mathrm{j}\sin(k\Omega_0 t)]$$

- Grouping the cos- and sin-terms gives

$$x(t) = X_0 + 2\sum_{k=1}^{\infty} \frac{X_k + X_{-k}}{2}\cos(k\Omega_0 t) + 2\mathrm{j}\sum_{k=1}^{\infty} \frac{X_k - X_{-k}}{2}\sin(k\Omega_0 t)$$

- Finally, we compute

$$\frac{X_k + X_{-k}}{2} = \frac{1}{2T_0}\int_{t=t_0}^{t_0+T_0} x(t)(e^{-jk\Omega_0 t} + e^{jk\Omega_0 t})\,dt$$
$$= \frac{1}{T_0}\int_{t=t_0}^{t_0+T_0} x(t)\cos(k\Omega_0 t)\,dt =: c_k$$

$$j\frac{X_k - X_{-k}}{2} = \frac{j}{2T_0}\int_{t=t_0}^{t_0+T_0} x(t)(e^{-jk\Omega_0 t} - e^{jk\Omega_0 t})\,dt$$
$$= \frac{1}{T_0}\int_{t=t_0}^{t_0+T_0} x(t)\sin(k\Omega_0 t)\,dt =: d_k$$

Trigonometric Fourier series

- In conclusion

$$x(t) = c_0 + 2\sum_{k=1}^{\infty} c_k \cos(k\Omega_0 t) + d_k \sin(k\Omega_0 t)$$

- with

$$c_k = \frac{X_k + X_{-k}}{2}, \quad k = 0, 1, 2, \ldots$$

and

$$d_k = \mathrm{j}\frac{X_k - X_{-k}}{2}, \quad k = 1, 2, \ldots .$$

- This is the *trigonometric Fourier series*

- Let $x(t)$ be a periodic signal with fundamental period T_0
- Consider a one-period restriction of this signal

$$x_1(t) = x(t)[u(t - t_0) - u(t - t_0 - T_0)]$$

- **Warning:** do not confuse this signal with the partial sum $x_1(t)$

- The Laplace transform of $x_1(t)$ is

$$X_1(s) = \int_{t=-\infty}^{\infty} x_1(t)e^{-st}\,dt = \int_{t=t_0}^{t_0+T_0} x(t)e^{-st}\,dt$$

- The Fourier expansion coefficient of $x(t)$ is given by

$$X_k = \frac{1}{T_0}\int_{t=t_0}^{t_0+T_0} x(t)e^{-jk\Omega_0 t}\,dt$$

- A comparison with the Laplace transform of $x_1(t)$ reveals that

$$X_k = \frac{1}{T_0}X_1(s)\Big|_{s=jk\Omega_0}, \quad k = 0, \pm 1, \pm 2, \ldots$$

Response of LTI systems to periodic signals

- Consider an LTI system with input signal $x(t)$, impulse response $h(t)$, and output signal $y(t)$
- We have
$$y(t) = \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau$$
- Finally, let $H(s)$ denote the transfer function of the LTI system
- Input signal $x(t)$: a periodic signal with fundamental period T_0
- What is the output?

Response of LTI systems to periodic signals

- Fourier expansion of $x(t)$: $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}$
- For the output signal we have

$$
\begin{aligned}
y(t) &= \int_{\tau=-\infty}^{\infty} h(\tau) x(t-\tau)\, d\tau \\
&= \int_{\tau=-\infty}^{\infty} h(\tau) \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 (t-\tau)}\, d\tau \\
&= \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t} \int_{\tau=-\infty}^{\infty} h(\tau) e^{-jk\Omega_0 \tau}\, d\tau \\
&= \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t} H(jk\Omega_0) = \sum_{k=-\infty}^{\infty} Y_k e^{jk\Omega_0 t}
\end{aligned}
$$

- with $Y_k = X_k H(jk\Omega_0)$

Response of LTI systems to periodic signals

- Output signal $y(t)$ is also periodic with fundamental period T_0 and its Fourier coefficients are given by

$$Y_k = X_k H(jk\Omega_0), \quad k = 0, \pm 1, \pm 2, \dots .$$