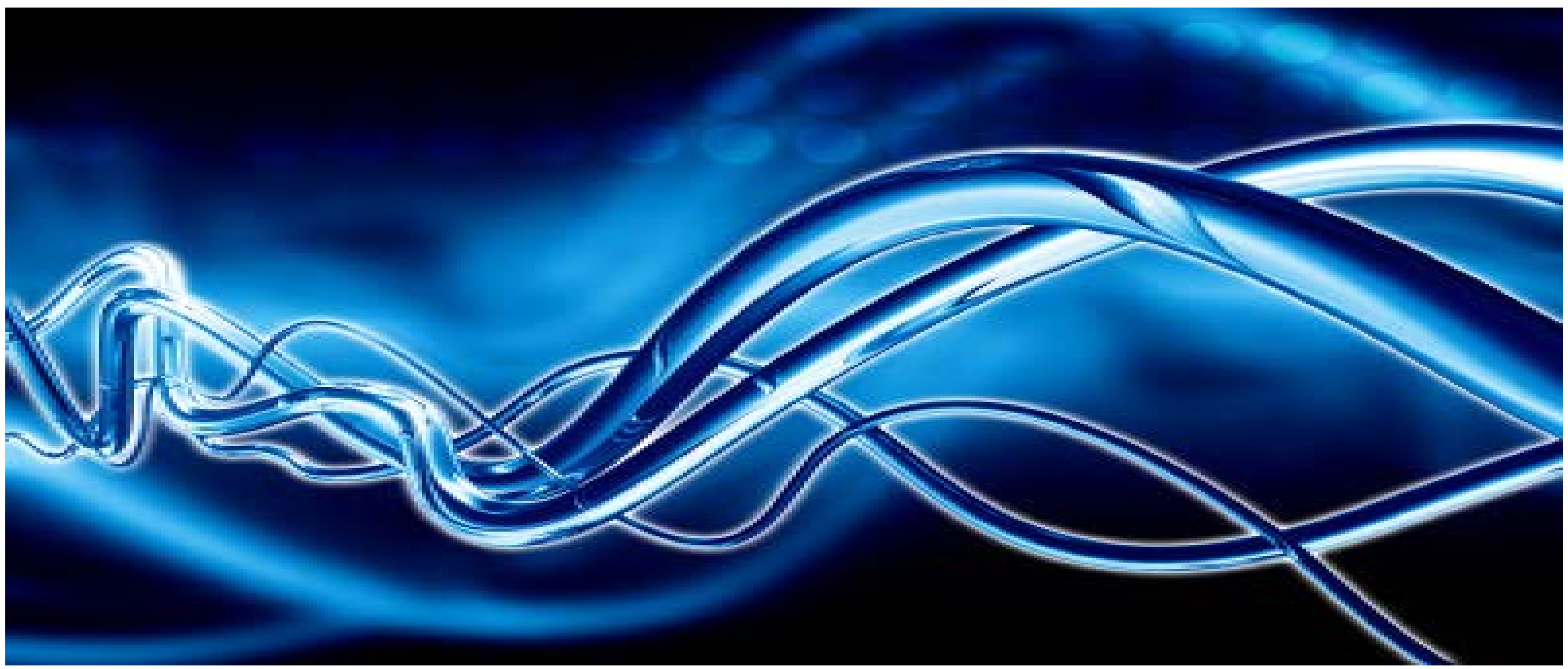
Signals and Systems



4. The Laplace Transform Part 2

The Inverse Laplace Transform

- **Contents**
  The inverse Laplace transform
  The steady-state response of an LTI system

- **Book:**
  Sections 3.5, 3.6, 3.7, 3.8

- **Exercises:**
  3.1, 3.3, 3.4, 3.6, 3.7, 3.8, 3.13, 3.15, 3.17, 3.20, 3.21 (3rd Ed.)
  3.2, 3.4, 3.5, 3.9, 3.10, 3.13, 3.20, 3.22, 3.25, 3.29, 3.30 (2nd Ed.)

The Inverse Laplace Transform

- The two-sided Laplace transform of a signal $x(t)$ is given by

$$X(s) = \int_{t=-\infty}^{\infty} x(t) e^{-st} \, \mathrm{d}t, \qquad s \in \mathrm{ROC}_x$$

- The correspondence between $X(s) + \mathrm{ROC}_x$ and $x(t)$ is unique
- Actually, we have already used this property without being very explicit about it (in the previous lecture we used Laplace transform tables)

The Inverse Laplace Transform

- We have an explicit expression for the Laplace transform of a signal $x(t)$ producing $X(s)$ along with its ROC
- Is there an explicit expression for the *inverse* transform?
- In other words, given $X(s)$ and its ROC, is there an explicit expression or operator that produces the time-signal $x(t)$?

The Inverse Laplace Transform

- The answer is yes
- We claim that the inverse Laplace transform is given by

$$x(t) = \frac{1}{2\pi \mathrm{j}} \int_{s=\sigma-\mathrm{j}\infty}^{\sigma+\mathrm{j}\infty} X(s) e^{st} \, \mathrm{d}s,$$

where the integration contour is located in ROC_x

The Inverse Laplace Transform

- This contour is called the *Bromwich contour*
- We also write

$$x(t) = \frac{1}{2\pi \mathrm{j}} \int_{s \in \mathrm{Br}} X(s) e^{st} \, \mathrm{d}s$$

with

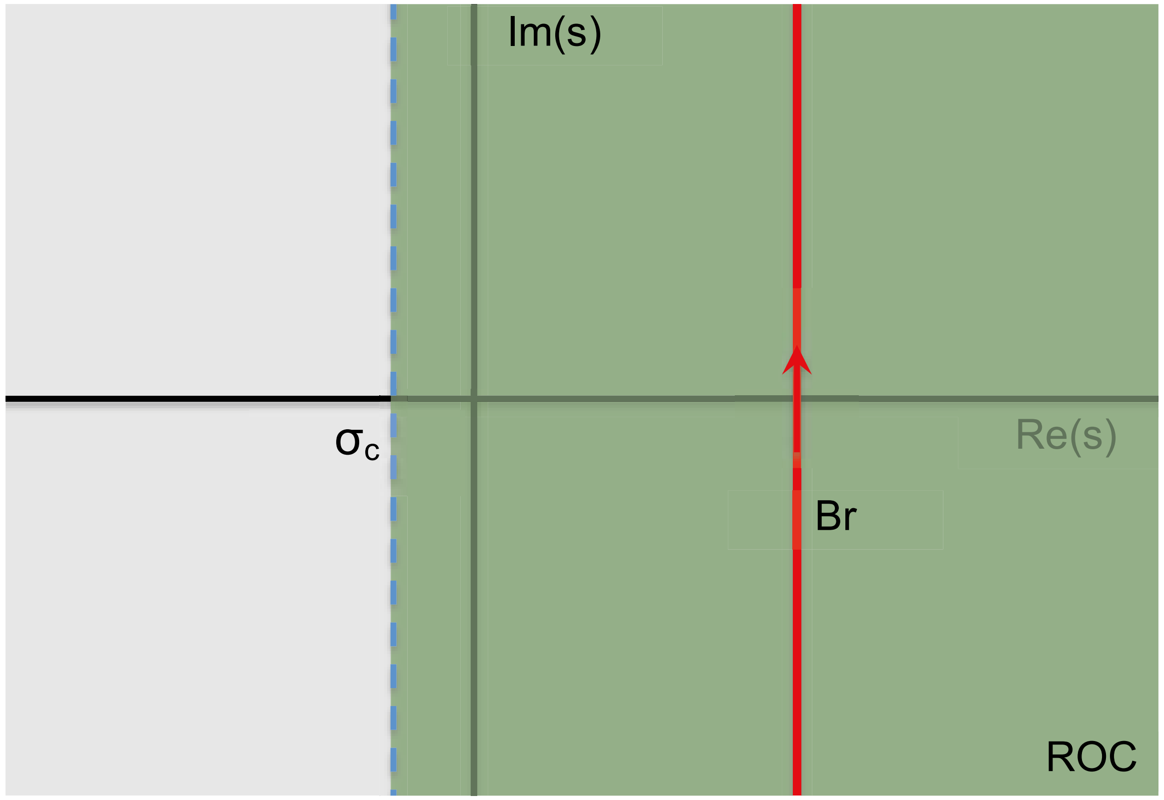
$$\mathrm{Br} = \{ s \in \mathrm{ROC}_x | s = \sigma + \mathrm{j}\Omega, -\infty < \Omega < \infty \}$$

The Inverse Laplace Transform

Thomas John I’Anson Bromwich
Born 1875
Died 1929

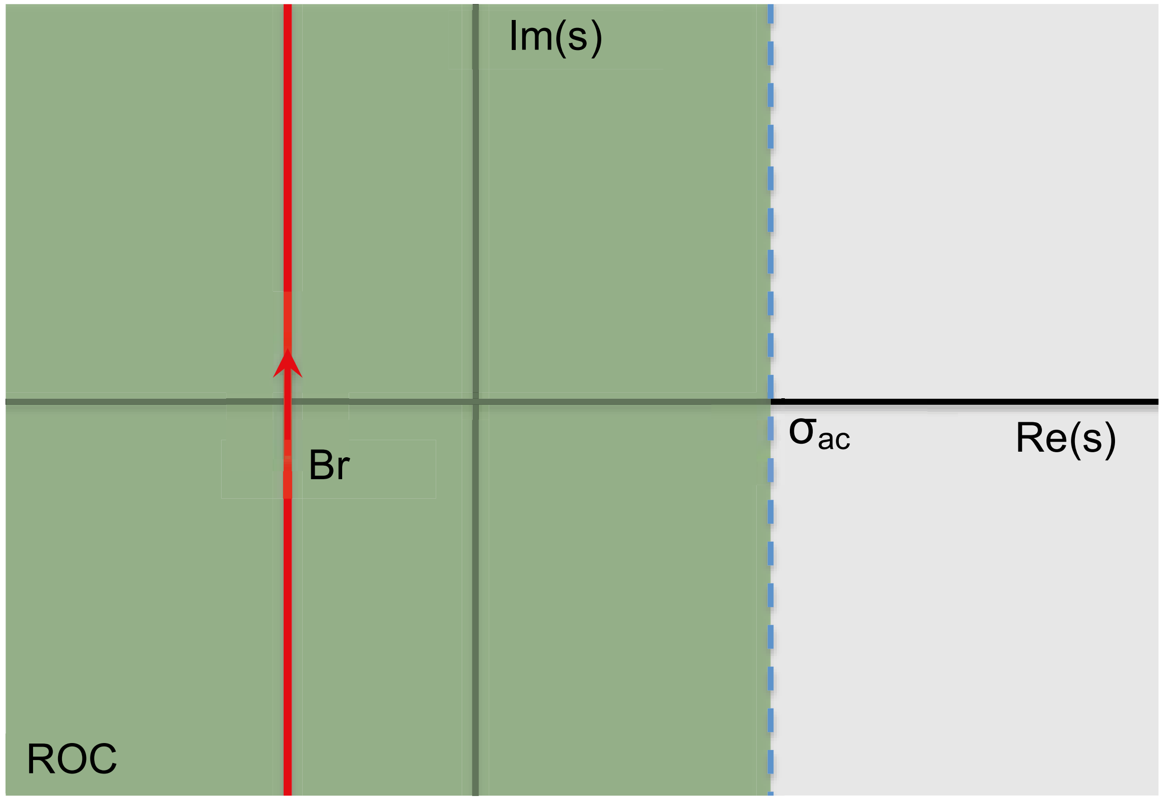
The Inverse Laplace Transform

- Assuming the Laplace transform $X(s)$ exists
- *Causal signals* $x(t)$: the Bromwich contour is located within some *right-half plane* = ROC_x



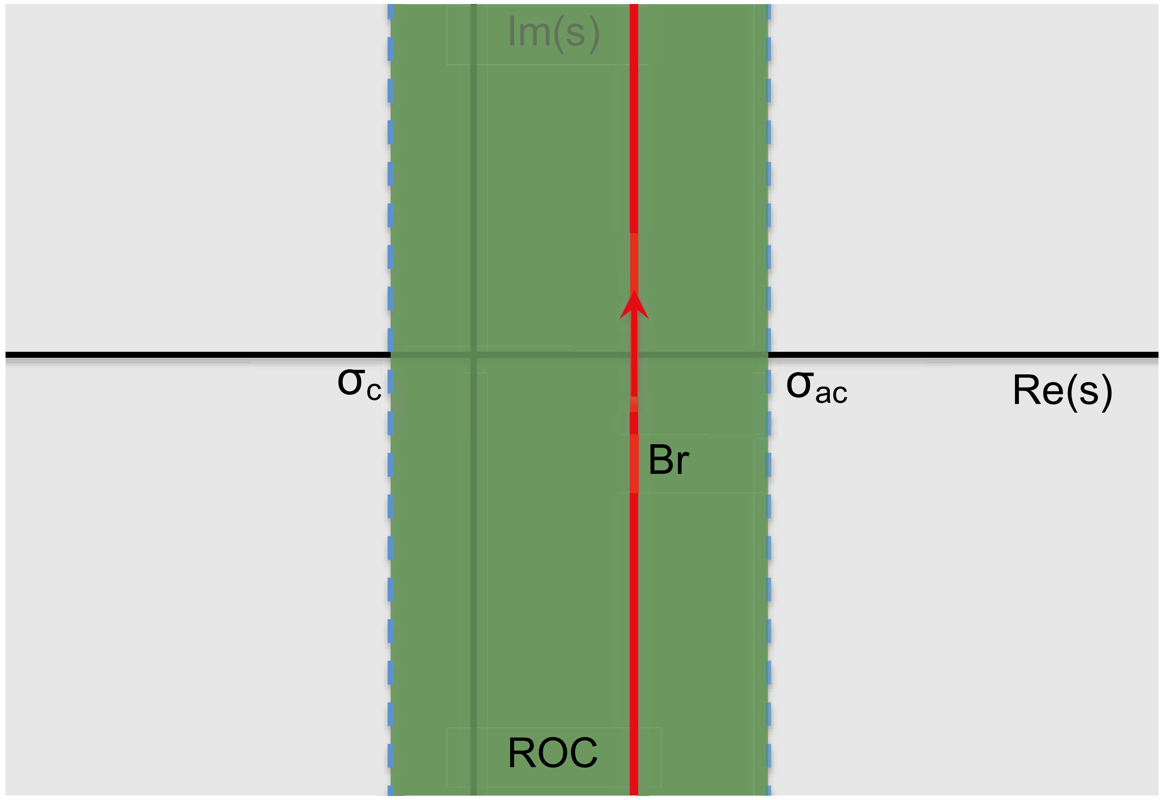
The Inverse Laplace Transform

- *Anti-causal signals* $x(t)$: the Bromwich contour is located within some *left-half plane* $= \mathrm{ROC}_x$



The Inverse Laplace Transform

- *Noncausal signals* $x(t)$: the Bromwich contour is located within some *strip* = ROC_x



The Inverse Laplace Transform

- Let us verify that the proposed inversion formula indeed produces the time-domain signal $x(t)$
- The inversion formula is given by

$$\frac{1}{2\pi \mathrm{j}} \int_{s \in \mathrm{Br}} X(s) e^{st} \, \mathrm{d}s \qquad \text{with } s \in \mathrm{ROC}_x$$

The Inverse Laplace Transform

- We start by substituting the expression for the Laplace transform in this formula
- We get

$$\frac{1}{2\pi \mathrm{j}} \int_{s \in \mathrm{Br}} X(s)\, e^{st}\, \mathrm{d}s = \frac{1}{2\pi \mathrm{j}} \int_{s \in \mathrm{Br}} \int_{\tau=-\infty}^{\infty} x(\tau) e^{-s\tau}\, \mathrm{d}\tau\, e^{st}\, \mathrm{d}s$$

- Interchanging the order of integration results in

$$\frac{1}{2\pi \mathrm{j}} \int_{s \in \mathrm{Br}} X(s)\, e^{st}\, \mathrm{d}s = \frac{1}{2\pi \mathrm{j}} \int_{\tau=-\infty}^{\infty} x(\tau) \int_{s=\sigma-\mathrm{j}\infty}^{\sigma+\mathrm{j}\infty} e^{s(t-\tau)}\, \mathrm{d}s\, \mathrm{d}\tau$$

The Inverse Laplace Transform

- Introducing a new variable of integration $p = s - \sigma$, we can write

$$\frac{1}{2\pi \mathrm{j}} \int_{s\in \mathrm{Br}} X(s)\, e^{st}\, \mathrm{d}s = \frac{1}{2\pi \mathrm{j}} \int_{\tau=-\infty}^{\infty} x(\tau) \int_{p=-\mathrm{j}\infty}^{\mathrm{j}\infty} e^{(p+\sigma)(t-\tau)}\, \mathrm{d}p\, \mathrm{d}\tau$$

$$= \frac{1}{2\pi \mathrm{j}} \int_{\tau=-\infty}^{\infty} x(\tau)\, e^{\sigma(t-\tau)} \int_{p=-\mathrm{j}\infty}^{\mathrm{j}\infty} e^{p(t-\tau)}\, \mathrm{d}p\, \mathrm{d}\tau$$

- With $p = \mathrm{j}\Omega$ ($\mathrm{d}p = \mathrm{j}\mathrm{d}\Omega$) this becomes

$$\frac{1}{2\pi \mathrm{j}} \int_{s\in \mathrm{Br}} X(s)\, e^{st}\, \mathrm{d}s = \int_{\tau=-\infty}^{\infty} x(\tau)\, e^{\sigma(t-\tau)} \left[\frac{1}{2\pi} \int_{\Omega=-\infty}^{\infty} e^{\mathrm{j}\Omega(t-\tau)}\, \mathrm{d}\Omega \right] \mathrm{d}\tau$$

The Inverse Laplace Transform

- Now recall the *completeness relation* from Lecture 1:

$$\delta(t) = \frac{1}{2\pi} \int_{\Omega=-\infty}^{\infty} e^{\mathrm{j}\Omega t}\, \mathrm{d}\Omega$$

- Consequently,

$$\delta(t-\tau) = \frac{1}{2\pi} \int_{\Omega=-\infty}^{\infty} e^{\mathrm{j}\Omega (t-\tau)}\, \mathrm{d}\Omega$$

The Inverse Laplace Transform

- Using this result, we obtain

$$\frac{1}{2\pi \mathrm{j}} \int_{s\in\mathrm{Br}} X(s)\, e^{st}\, \mathrm{d}s = \int_{\tau=-\infty}^{\infty} x(\tau)\, e^{\sigma(t-\tau)}\, \delta(t-\tau)\, \mathrm{d}\tau = x(t)$$

- **Laplace transformation pair**
- Forward transformation:

$$X(s) = \int_{t=-\infty}^{\infty} x(t) e^{-st}\, \mathrm{d}t, \qquad s \in \mathrm{ROC}_x$$

- Inverse transformation:

$$x(t) = \frac{1}{2\pi \mathrm{j}} \int_{s\in\mathrm{Br}} X(s) e^{st}\, \mathrm{d}s, \qquad \mathrm{Br} \in \mathrm{ROC}_x$$

The Inverse Laplace Transform

- If the imaginary axis is contained in ROC_x then we can restrict the Laplace parameter to the imaginary axis

- Setting $s = \text{j}\Omega$, the Laplace transformation pair becomes

The Inverse Laplace Transform

- Forward transformation:

$$X(\Omega) = \int_{t=-\infty}^{\infty} x(t) e^{-j\Omega t}\, dt$$

- Inverse transformation:

$$x(t) = \frac{1}{2\pi} \int_{\Omega=-\infty}^{\infty} X(\Omega) e^{j\Omega t}\, d\Omega$$

- This transformation pair defines the *Fourier transformation* (much more on this later)

The Inverse Laplace Transform

- **Warning!** The Fourier transformation defined above is according to the convention used by electrical engineers
- Physicists use the letter i for the imaginary unit and take $s = -\mathrm{i}\Omega$ in the Laplace transform

The Inverse Laplace Transform

- The Fourier transformation pair of a physicist is
- Forward transformation:

$$X(\Omega) = \int_{t=-\infty}^{\infty} x(t) e^{\mathrm{i}\Omega t} \, \mathrm{d}t$$

- Inverse transformation:

$$x(t) = \frac{1}{2\pi} \int_{\Omega=-\infty}^{\infty} X(\Omega) e^{-\mathrm{i}\Omega t} \, \mathrm{d}\Omega$$

- When reading books, papers, reports, etc. check out the convention that the author uses

The Inverse Laplace Transform

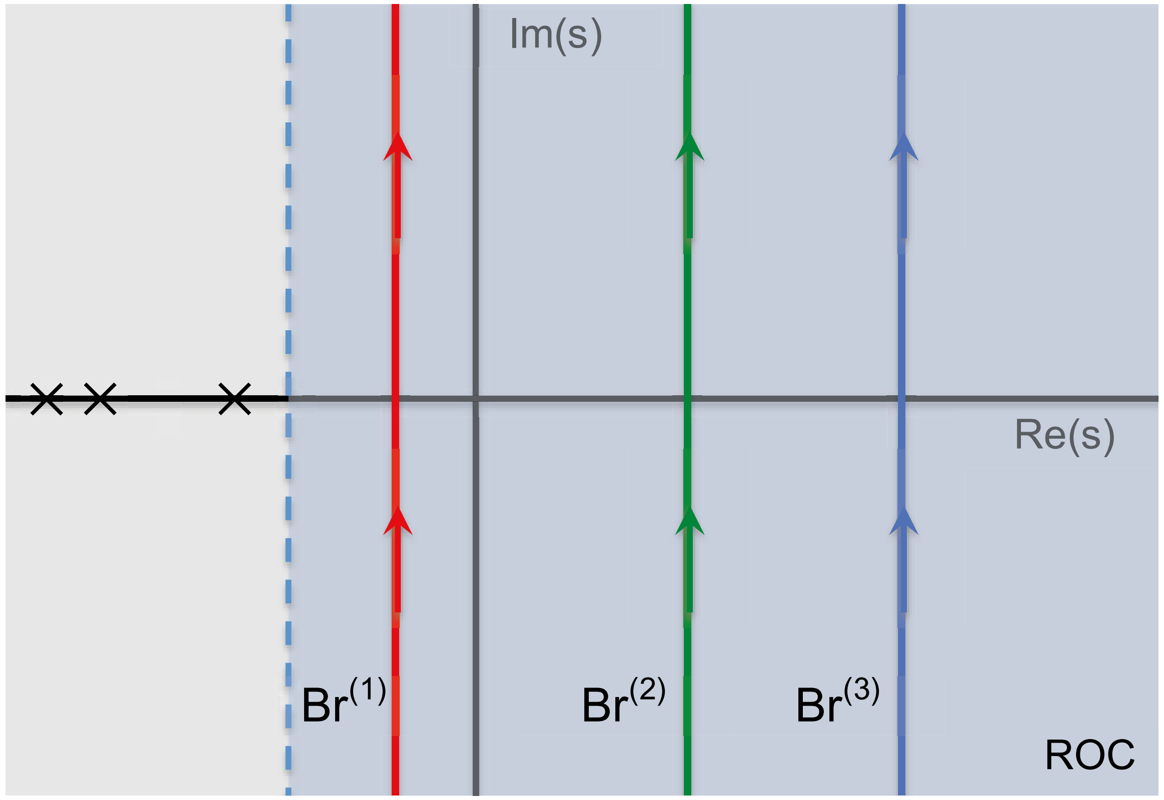
- BIBO stability is related to the existence of the Fourier transform
- Specifically, if the system is BIBO stable then $H(\Omega)$ exists

$$|H(\Omega)| = \left|\int_{t=-\infty}^{\infty} h(t)e^{-j\Omega t}\,\mathrm{d}t\right| \leq \int_{t=-\infty}^{\infty} \left|h(t)e^{-j\Omega t}\right|\mathrm{d}t \leq \int_{t=-\infty}^{\infty} |h(t)|\,\mathrm{d}t < \infty$$

- The converse (existence of $H(\Omega)$ implies BIBO stability) is true under certain conditions
- More on this later (see slides 94 – 96)

The Inverse Laplace Transform

- Returning to the Laplace transform, we note that *any* Bromwich contour in the inverse Laplace transform does the job so long as it belongs to the ROC



- To see this, we recall Cauchy's theorem from complex analysis
- Loosely speaking, this theorem states that if $F(s)$ is analytic within a region $\mathscr{A}$ in the complex s-plane, then for any closed curve C belonging to $\mathscr{A}$, we have

$$\oint_{s \in C} F(s) \mathrm{d}s = 0$$

The Inverse Laplace Transform

- For a time signal $x(t)$ we know that its Laplace transform $X(s)$ is analytic in its ROC
- The function e^{st} is also analytic in this region (as a function of s)
- Conclusion: the function $F(s) = X(s)e^{st}$ is analytic in the ROC of the signal $x(t)$

The Inverse Laplace Transform

- Applying Cauchy's theorem, we have

$$\oint_{s \in C} X(s) e^{st} \mathrm{d}s = 0$$

for any closed curve C belong to the ROC of the signal $x(t)$

The Inverse Laplace Transform

- This result can be used to show that integration along any Bromwich contour belonging to the ROC produces the time-domain signal $x(t)$
- We illustrate this for a causal time-signal $x(t)$ (the analysis for anti- or non-causal signals is similar)
- For a causal time signal, the ROC is some right-half plane in general
- We consider two Bromwich contours Br_1 and Br_2 belonging to this region

The Inverse Laplace Transform

- Our claim is that it does not matter along which contour you integrate to get $x(t)$ back
- In other words

$$\int_{s\in \mathrm{Br}_1} X(s)e^{st}\,\mathrm{d}s = \int_{s\in \mathrm{Br}_2} X(s)e^{st}\,\mathrm{d}s$$

with

$$\mathrm{Br}_1 = \{s \in \mathbb{C} | s = \sigma_1 + \mathrm{j}\Omega, \sigma_1 > \sigma_\mathrm{c}, -\infty < \Omega < \infty\}$$

and

$$\mathrm{Br}_2 = \{s \in \mathbb{C} | s = \sigma_2 + \mathrm{j}\Omega, \sigma_2 > \sigma_\mathrm{c}, -\infty < \Omega < \infty\}$$

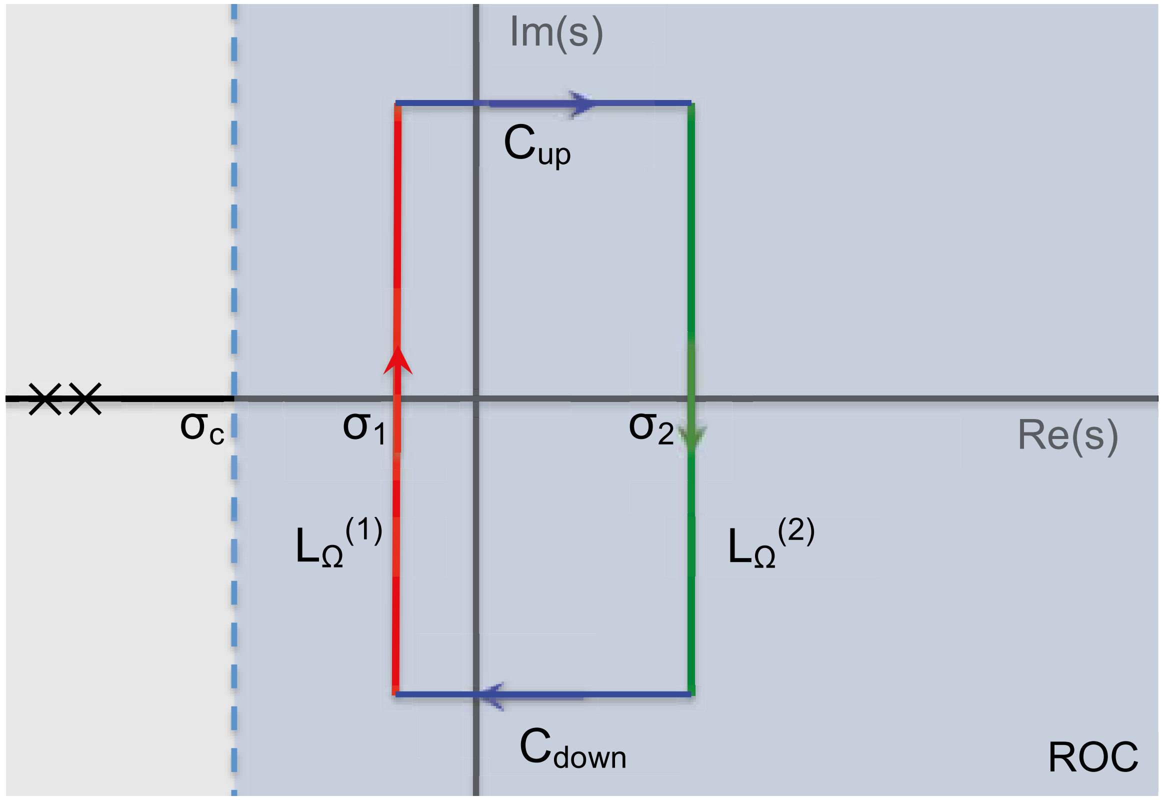
$\sigma_2 > \sigma_1$

The Inverse Laplace Transform

- To show this, consider the curve

$$C_\Omega = L_\Omega^{(1)} \cup C_{\text{up}} \cup L_\Omega^{(2)} \cup C_{\text{down}}$$

which is completely located within the ROC of signal $x(t)$



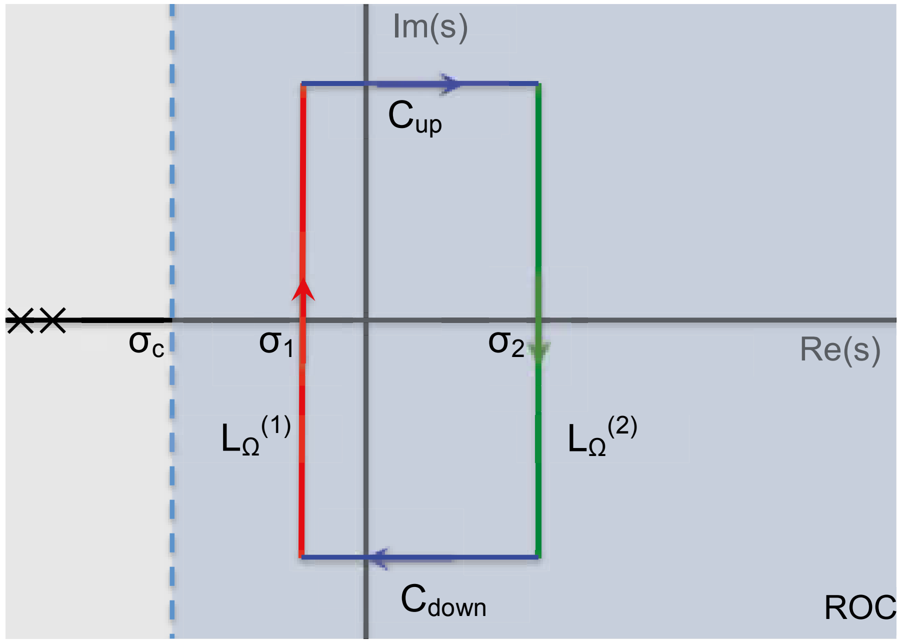
The Inverse Laplace Transform

- From Cauchy's theorem it follows that

$$\oint_{s\in C_\Omega} X(s)e^{st}\,\mathrm{d}s = 0 \qquad (*)$$

or

$$\int_{s\in L_\Omega^{(1)}} X(s)e^{st}\,\mathrm{d}s + \int_{s\in C_{\mathrm{up}}} X(s)e^{st}\,\mathrm{d}s + \int_{s\in L_\Omega^{(2)}} X(s)e^{st}\,\mathrm{d}s + \int_{s\in C_{\mathrm{down}}} X(s)e^{st}\,\mathrm{d}s = 0$$



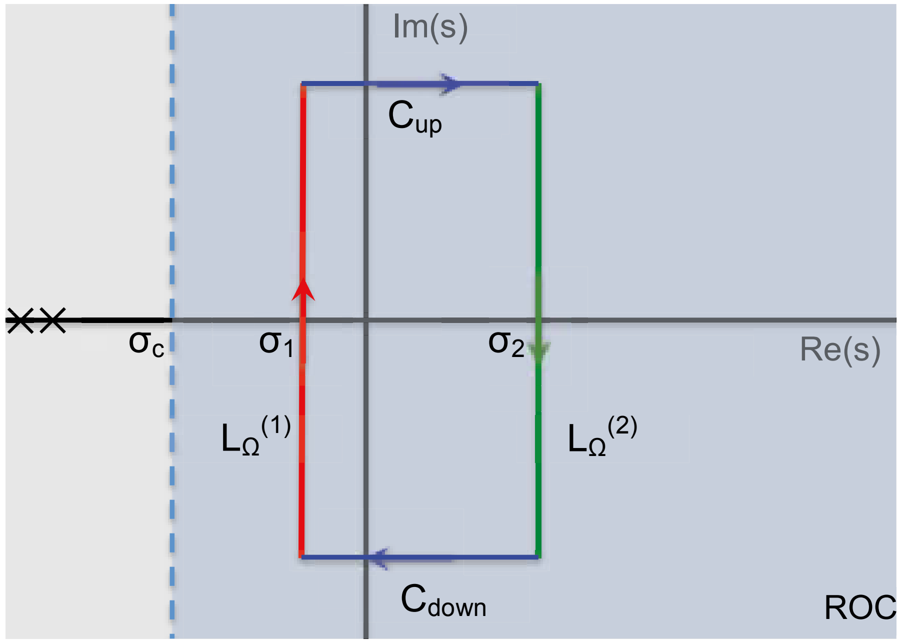
The Inverse Laplace Transform

- Clearly,

$$\lim_{\Omega\to\infty}\int_{s\in L_\Omega^{(1)}} X(s)e^{st}\,\mathrm{d}s = \int_{s\in \mathrm{Br}_1} X(s)e^{st}\,\mathrm{d}s$$

and

$$\lim_{\Omega\to\infty}\int_{s\in L_\Omega^{(2)}} X(s)e^{st}\,\mathrm{d}s = -\int_{s\in \mathrm{Br}_2} X(s)e^{st}\,\mathrm{d}s$$



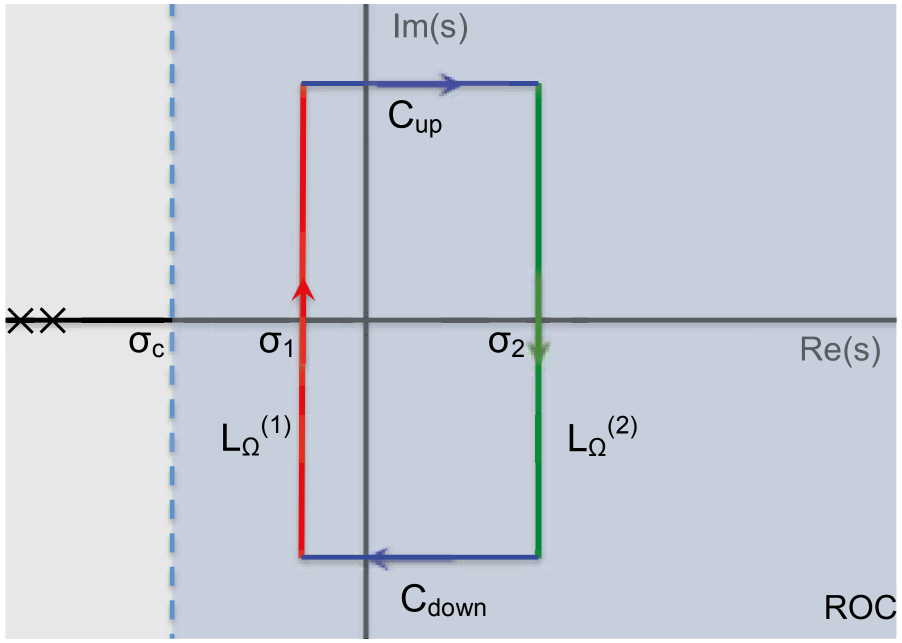
The Inverse Laplace Transform

- Furthermore, it can be shown that

$$\lim_{\Omega\to\infty}\int_{s\in C_{\text{up}}} X(s)e^{st}\,\mathrm{d}s = 0$$

and

$$\lim_{\Omega\to\infty}\int_{s\in C_{\text{down}}} X(s)e^{st}\,\mathrm{d}s = 0$$

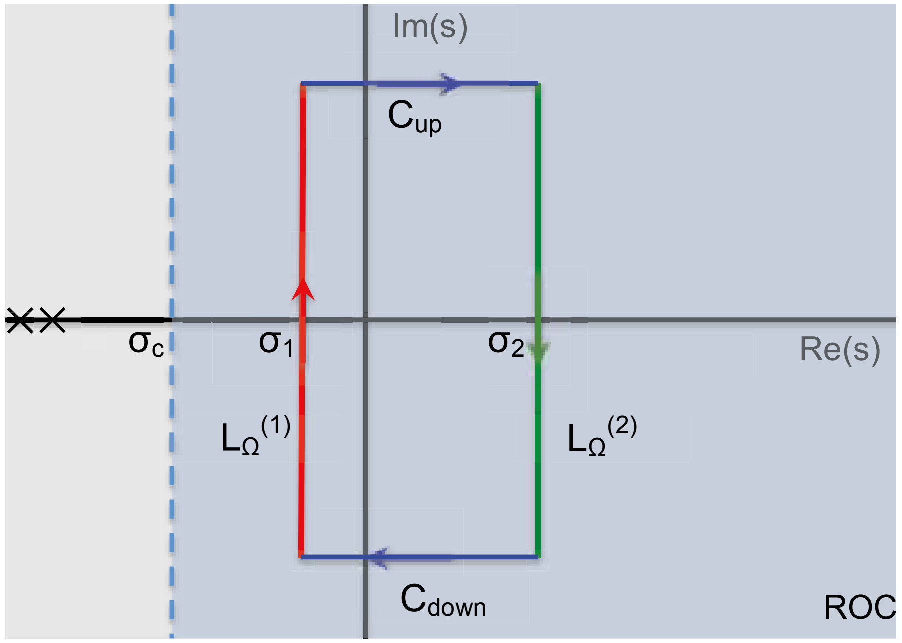


The Inverse Laplace Transform

- Taking the limit $\Omega \to \infty$ in Eq. $(*)$ and putting all limits together, we find

$$\int_{s \in \mathrm{Br}_1} X(s) e^{st} \, \mathrm{d}s - \int_{s \in \mathrm{Br}_2} X(s) e^{st} \, \mathrm{d}s = 0$$

which is what we wanted to show



The Inverse Laplace Transform

- To determine the inverse Laplace transform of some s-domain function $X(s)$, we continu the integrand of the inverse transform into the complex s-plane and use techniques from complex analysis

- Although the approach that we follow can be applied to a wide class of Laplace-domain functions $X(s)$, we restrict ourselves to cases where $X(s)$ is a *strictly proper* rational function of the form

$$X(s) = \frac{p_M(s)}{q_N(s)}$$

  with $p_M(s)$ is a polynomial in s of degree M and $q_N(s)$ is a polynomial in s of degree N

The Inverse Laplace Transform

- The rational function $X(s)$ is called *improper* if $M > N$
- The rational function $X(s)$ is called *proper* if $M \leq N$
- The rational function $X(s)$ is called *strictly proper* if $M < N$

The Inverse Laplace Transform

- **Examples**

$$X(s) = \frac{s^3 + 4}{s^2 + 1} \qquad \text{is improper}$$

$$X(s) = \frac{s}{s + 1} \qquad \text{is proper}$$

and

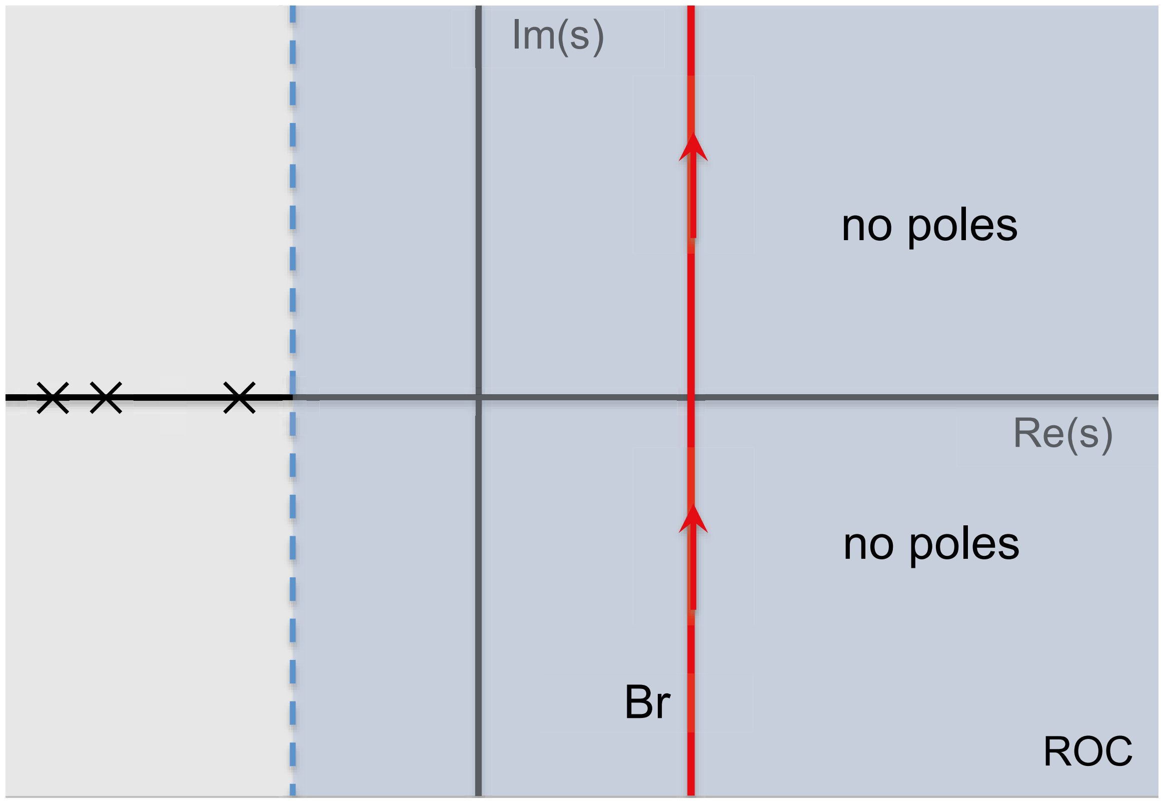
$$X(s) = \frac{s}{s^2 + 1} \qquad \text{is strictly proper}$$

The Inverse Laplace Transform

- Furthermore, let $X(s)$ have
  * m poles located to the *left* of the Bromwich contour and
  * n poles located to the *right* of the Bromwich contour

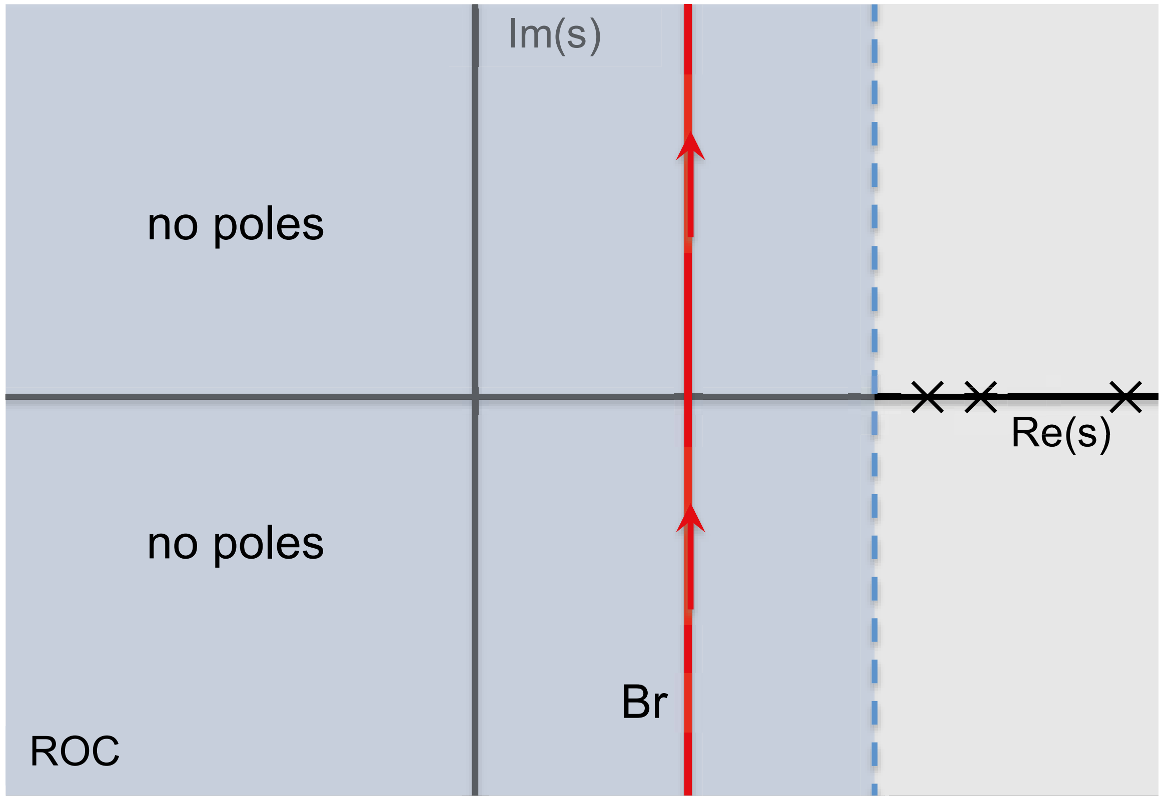
The Inverse Laplace Transform

- If $x(t)$ is causal then $n = 0$: there are no poles to the right of the Bromwich contour, since for a causal signal $X(s)$ is analytic to the right of the Bromwich contour



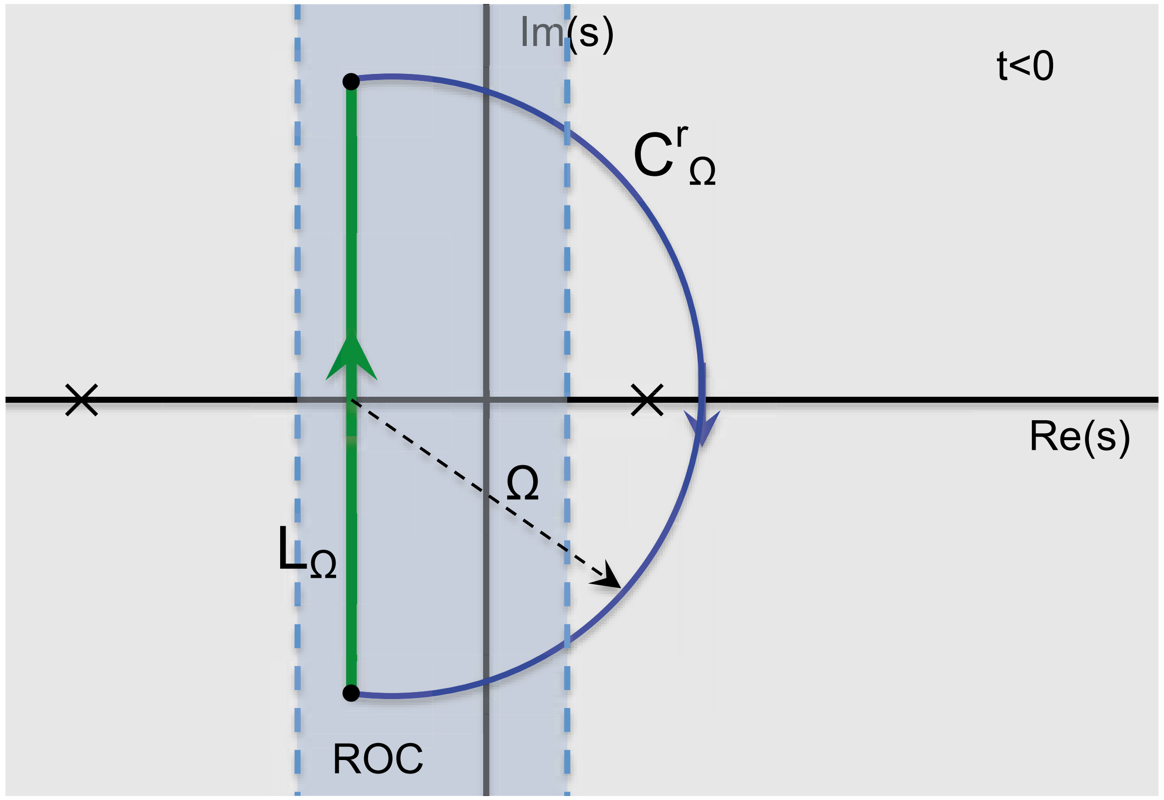
The Inverse Laplace Transform

- If $x(t)$ is anti-causal then $m = 0$: there are no poles to the left of the Bromwich contour, since for an anti-causal signal $X(s)$ is analytic to the left of the Bromwich contour



The Inverse Laplace Transform

- To evaluate the inversion integral, we distinguish between two cases
- **Case 1:** $t < 0$
- In this case we evaluate the integral by considering the closed curve $C_\Omega = L_\Omega \cup C^r_\Omega$ shown below



The Inverse Laplace Transform

- The curve C_Ω is traversed *clockwise* and encloses all n poles of $X(s)$ located to the right of the Bromwich contour
- We can always achieve this by making Ω sufficiently large
- Applying the residue theorem, we find

$$\oint_{s\in C_\Omega} X(s)e^{st}\,\mathrm{d}s = -2\pi\mathrm{j}\sum_{p=1}^{n}\mathrm{Res}\left[X(s)e^{st}, s_p\right] \qquad (**)$$

where s_p is the pth pole located to the right of the Bromwich contour

The Inverse Laplace Transform

- Recall that the residue of $X(s)e^{st}$ at a pole of order k at $s = s_p$ is computed as follows:

  1. Construct the function $\varphi(s) = (s - s_p)^k X(s)e^{st}$
  2. The residue of $X(s)e^{st}$ at $s = s_p$ is given by

$$\text{Res}\left[X(s)e^{st}, s_p\right] = \left.\frac{\varphi^{(k-1)}(s)}{(k-1)!}\right|_{s=s_p}$$

The Inverse Laplace Transform

- The reason for considering the indicated curve C_Ω is that for the Laplace-domain functions $X(s)$ considered here (strictly proper rational functions), it can be shown that

$$\lim_{\Omega\to\infty}\int_{s\in C_\Omega^{\mathrm{r}}} X(s)e^{st}\,\mathrm{d}s = 0 \qquad \text{for } t<0$$

- Taking the limit $\Omega \to \infty$ in Eq. $(**)$ and realizing that

$$\lim_{\Omega\to\infty}\int_{s\in L_\Omega} X(s)e^{st}\,\mathrm{d}s = \int_{s\in \mathrm{Br}} X(s)e^{st}\,\mathrm{d}s$$

we find that

$$\int_{s\in \mathrm{Br}} X(s)e^{st}\,\mathrm{d}s = -2\pi\mathrm{j}\sum_{p=1}^{n}\mathrm{Res}\left[X(s)e^{st}, s_p\right] \qquad \text{for } t<0$$

The Inverse Laplace Transform

- Consequently,

$$x(t) = -\sum_{p=1}^{n} \text{Res}\left[X(s)e^{st}, s_p\right] \qquad \text{for } t < 0$$

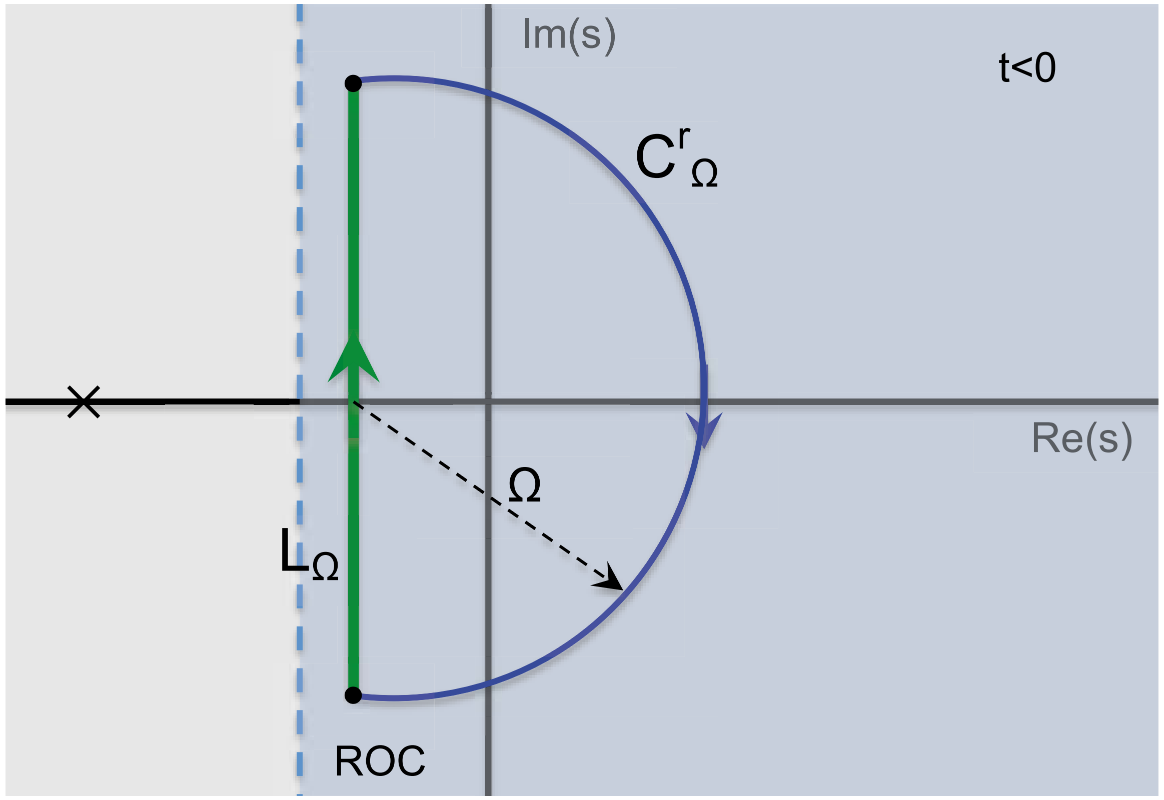
where the s_p are the distinct poles of $X(s)$ located to the right of the Bromwich contour

The Inverse Laplace Transform

- For a causal signal, $X(s)$ has no poles to the right of the Bromwich contour and the inversion formula gives

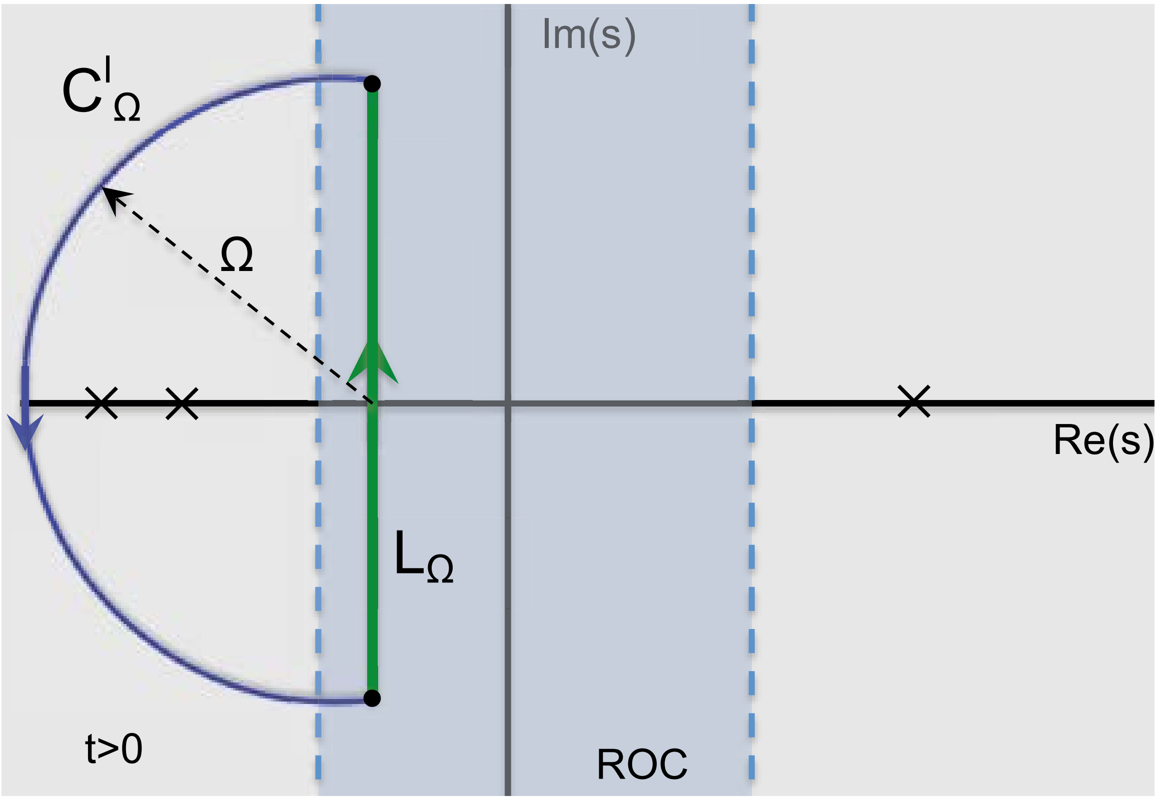
$$x(t) = 0 \qquad \text{for } t < 0$$

as it should be, of course



The Inverse Laplace Transform

- **Case 2:** $t > 0$
- In this case we evaluate the integral by considering the closed curve $C_\Omega = L_\Omega \cup C^{l}_{\Omega}$ shown below



- The curve C_Ω is traversed *counterclockwise* and encloses all m poles of $X(s)$ located to the left of the Bromwich contour
- We can always achieve this by making Ω sufficiently large
- Applying the residue theorem, we find

$$\oint_{s \in C_\Omega} X(s) e^{st}\, \mathrm{d}s = 2\pi \mathrm{j} \sum_{p=1}^{m} \mathrm{Res}\left[X(s)e^{st}, s_p\right] \qquad (***)$$

where s_p is the pth pole located to the left of the Bromwich contour

The Inverse Laplace Transform

- The reason for considering the indicated curve C_Ω is that for the Laplace-domain functions $X(s)$ consider here (strictly proper rational functions), it can be shown that

$$\lim_{\Omega\to\infty}\int_{s\in C_\Omega^1} X(s)e^{st}\,\mathrm{d}s = 0 \qquad \text{for } t > 0$$

The Inverse Laplace Transform

- Taking the limit $\Omega \to \infty$ in Eq. $(***)$ and realizing that

$$\lim_{\Omega\to\infty} \int_{s\in L_\Omega} X(s)e^{st}\,\mathrm{d}s = \int_{s\in \mathrm{Br}} X(s)e^{st}\,\mathrm{d}s$$

we find that

$$\int_{s\in \mathrm{Br}} X(s)e^{st}\,\mathrm{d}s = 2\pi \mathrm{j} \sum_{p=1}^{m} \mathrm{Res}\left[X(s)e^{st}, s_p\right] \qquad \text{for } t > 0$$

The Inverse Laplace Transform

- Consequently,

$$x(t) = \sum_{p=1}^{m} \text{Res}\left[X(s)e^{st}, s_p\right] \qquad \text{for } t > 0$$

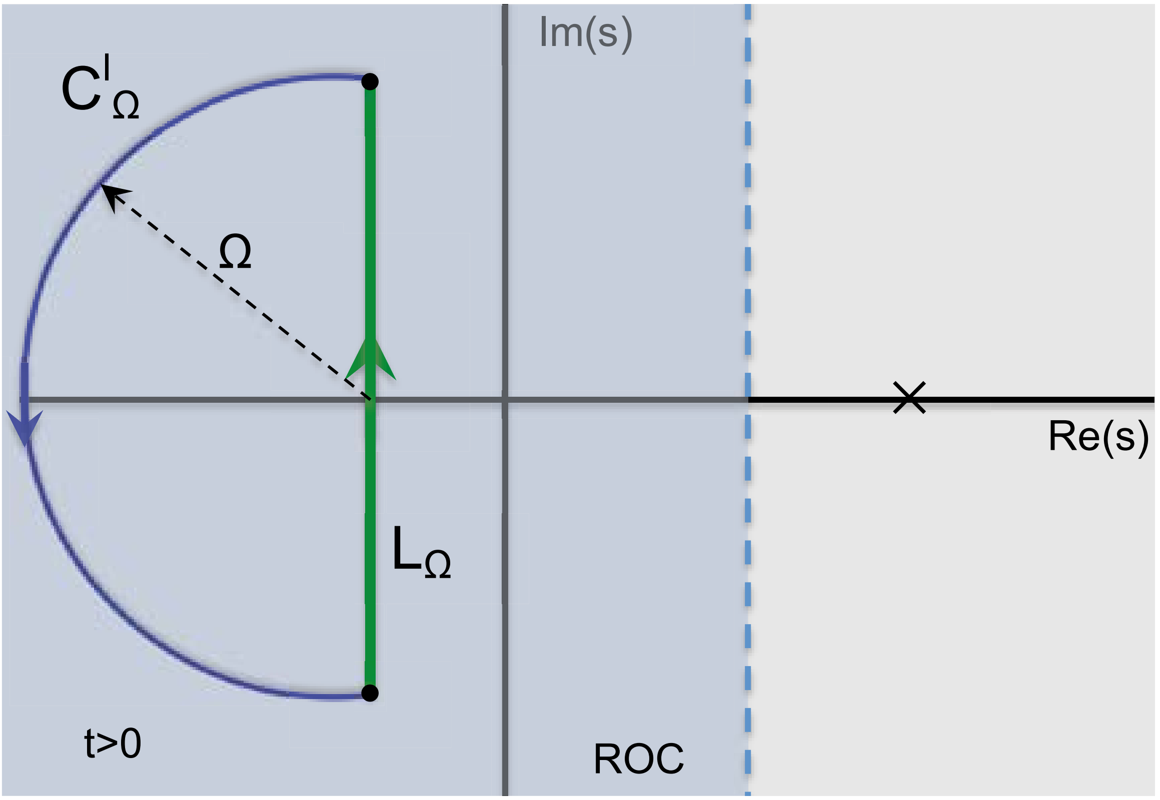
where the s_p are the distinct poles of $X(s)$ located to the left of the Bromwich contour

The Inverse Laplace Transform

- For an anti-causal signal, $X(s)$ has no poles to the left of the Bromwich contour and the inversion formula gives

$$x(t) = 0 \qquad \text{for } t > 0$$

as it should be, of course

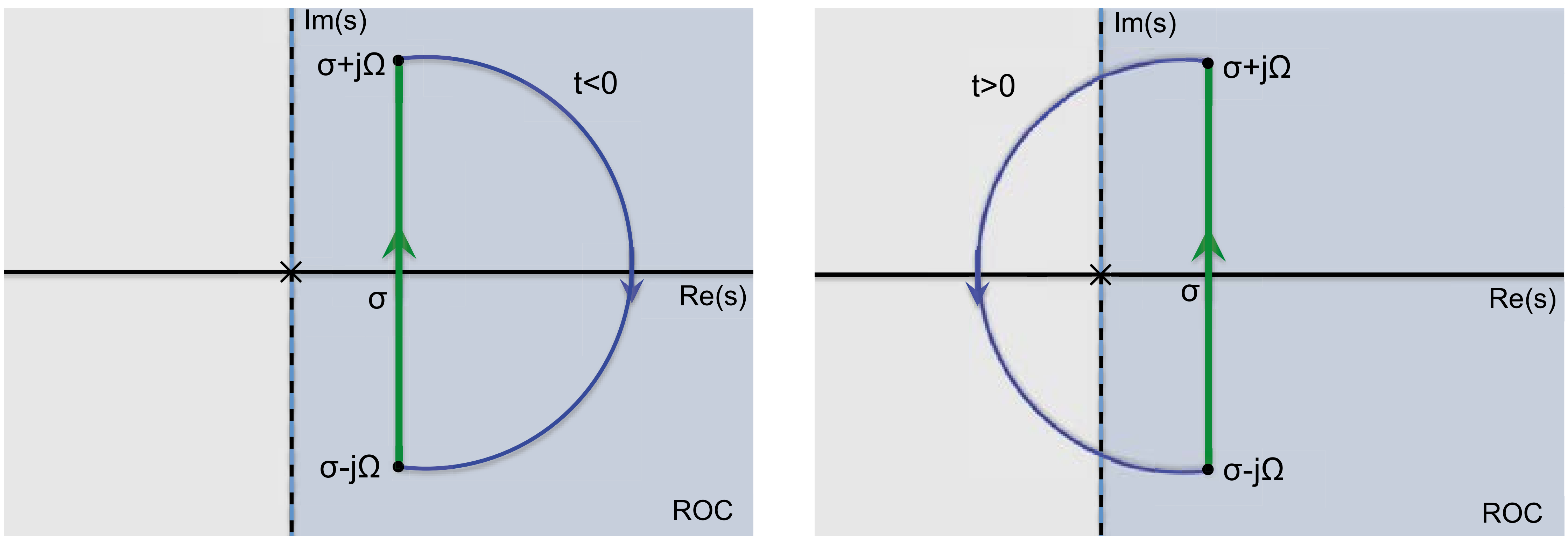


The Inverse Laplace Transform

- **Example 1** Let $X(s) = 1/s$ be the Laplace transform of a time signal $x(t)$ with the half-plane $\text{Re}(s) > 0$ as its ROC
- We already know what the time-function is, of course, but let's compute it using residue calculus
- $X(s)$ has a simple pole at $s = 0$ and is analytic on its ROC
- The Bromwich contour must be located within the ROC

The Inverse Laplace Transform

- Since there are no poles to the right of the Bromwich contour, we find $x(t) = 0$ for $t < 0$
- The simple pole at $s = 0$ is located to the left of the Bromwich contour outside the ROC, of course



The Inverse Laplace Transform

- Computing its residue, we find

$$\varphi(s) = sX(s)e^{st} = e^{st} \quad \text{and} \quad \text{Res}\left[\frac{e^{st}}{s}, 0\right] = \frac{\varphi(s)}{0!}\Big|_{s=0} = \frac{1}{1} = 1$$

- and the time signal is

$$x(t) = 1 \qquad \text{for } t > 0$$

- Conclusion: $x(t) = u(t)$

The Inverse Laplace Transform

- What happens at $t = 0$?
- Using the inversion formula, we find

$$x(0) = \frac{1}{2\pi \mathrm{j}} \int_{s \in \mathrm{Br}} \frac{1}{s} \, \mathrm{d}s = \lim_{\substack{\Omega_1 \to \infty \\ \Omega_2 \to \infty}} \int_{\sigma - \mathrm{j}\Omega_1}^{\sigma + \mathrm{j}\Omega_2} \frac{1}{s} \, \mathrm{d}s$$

- By changing the ratio Ω_1/Ω_2 we can give the integral any value that we want
- Setting $\Omega_1/\Omega_2 = 1$ (as is usual), the resulting integral is known as a *Cauchy principal value* integral

The Inverse Laplace Transform

- With this choice, we have

$$
\begin{aligned}
x(0) &= \frac{1}{2\pi \mathrm{j}} \int_{s\in \mathrm{Br}} \frac{1}{s}\,\mathrm{d}s = \frac{1}{2\pi \mathrm{j}} \lim_{\Omega\to\infty} \Big[\ln|s| + \mathrm{j}\arg(s) \Big]_{s=\sigma-\mathrm{j}\Omega}^{\sigma+\mathrm{j}\Omega} \\
&= \frac{1}{2\pi \mathrm{j}} \cdot 2\mathrm{j} \cdot \lim_{\Omega\to\infty} \arctan\left(\frac{\Omega}{\sigma}\right) \\
&= \frac{1}{2\pi \mathrm{j}} \cdot 2\mathrm{j} \cdot \frac{\pi}{2} = \frac{1}{2}
\end{aligned}
$$

The Inverse Laplace Transform

- For this reason, the Heaviside unit step function is often defined as

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{2} & \text{for } t = 0 \\ 1 & \text{for } t > 0 \end{cases}$$

- The above result can be generalized to a general discontinuous signals
- We have

$$\frac{x(t+0) + x(t-0)}{2} = \frac{1}{2\pi \mathrm{j}} \int_{s \in \mathrm{Br}} X(s) e^{st} \, \mathrm{d}s$$

The Inverse Laplace Transform

- **Example 2** Again $X(s) = 1/s$, but this time the ROC is $\{s \in \mathbb{C} | \mathrm{Re}(s) < 0\}$
- The ROC is now a left-half plane
- The Bromwich contour is located inside the ROC
- There are no poles to the left of the Bromwich contour
- Consequently,

$$x(t) = 0 \qquad \text{for } t > 0$$

The Inverse Laplace Transform

- The simple pole at $s = 0$ is now located to the right of the Bromwich contour and contributes for $t < 0$
- Using the residue formula for $t < 0$, we find

$$x(t) = -1 \qquad \text{for } t < 0$$

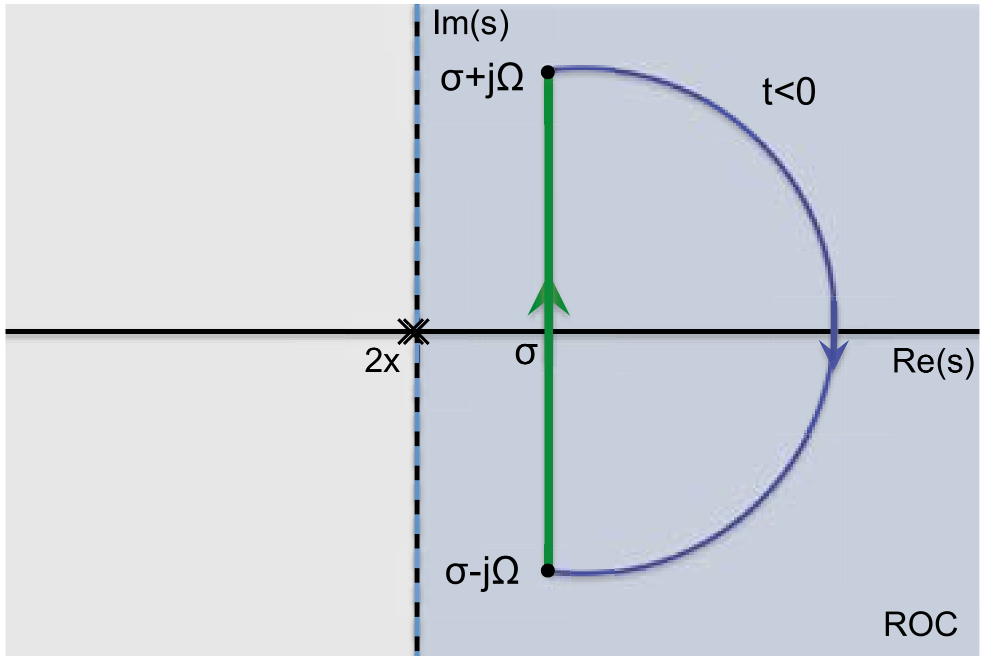
Don't forget the minus sign!

- In total: $x(t) = -u(-t)$

The Inverse Laplace Transform

- **Example 3** Suppose $X(s) = 1/s^2$ with $\text{Re}(s) > 0$ as its ROC
- What is the corresponding time signal?
- The Bromwich contour must be located within the ROC
- There are no poles to the right of the Bromwich contour
- Consequently,

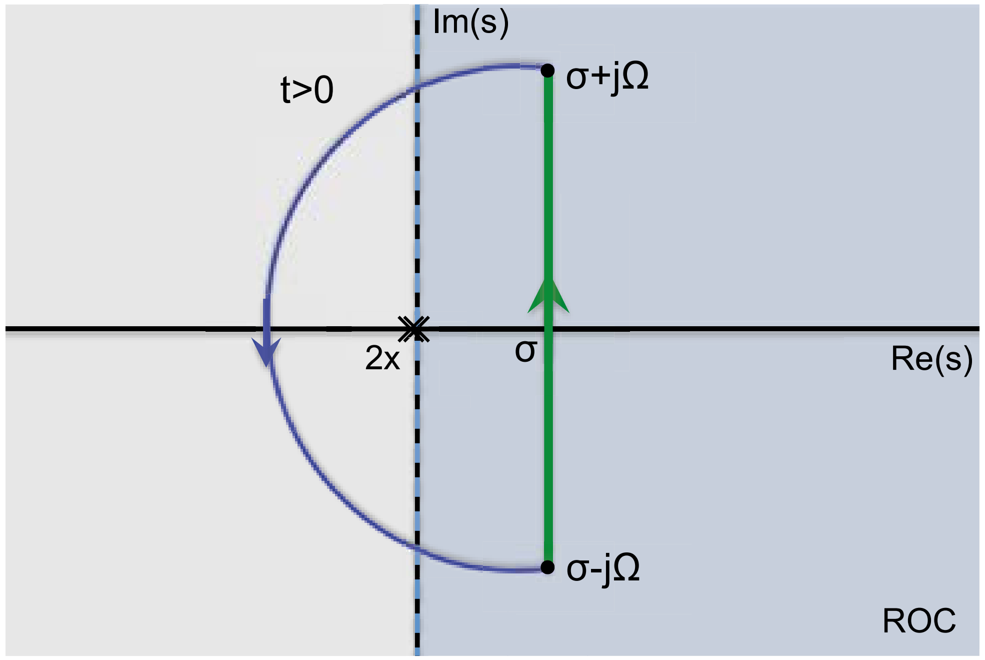
$$x(t) = 0 \qquad \text{for } t < 0$$



The Inverse Laplace Transform

- For $t > 0$ we encounter a pole of order 2 at the origin
- We compute its residue
- First, construct $\varphi(s)$:

$$\varphi(s) = s^2 X(s) e^{st} = e^{st}$$



- The residue at $s = 0$ is given by

$$\mathrm{Res}\left[X(s)e^{st}, 0\right] = \frac{\varphi^{(1)}(s)}{1!}\Big|_{s=0}$$

- Computing the derivative gives

$$\varphi^{(1)}(s) = \frac{\mathrm{d}}{\mathrm{d}s}e^{st} = te^{st}$$

The Inverse Laplace Transform

- and the residue is found as

$$\text{Res}\left[X(s)e^{st},0\right] = \left.\frac{te^{st}}{1!}\right|_{s=0} = t$$

- Substitution in the residue formula for $t > 0$ gives

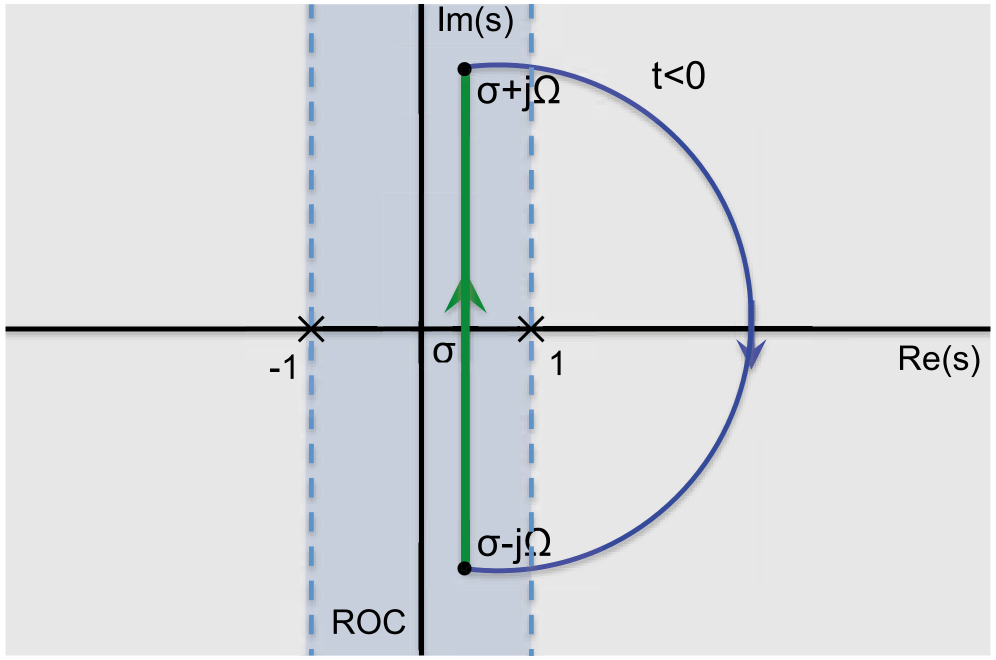
$$x(t) = t \qquad \text{for } t > 0$$

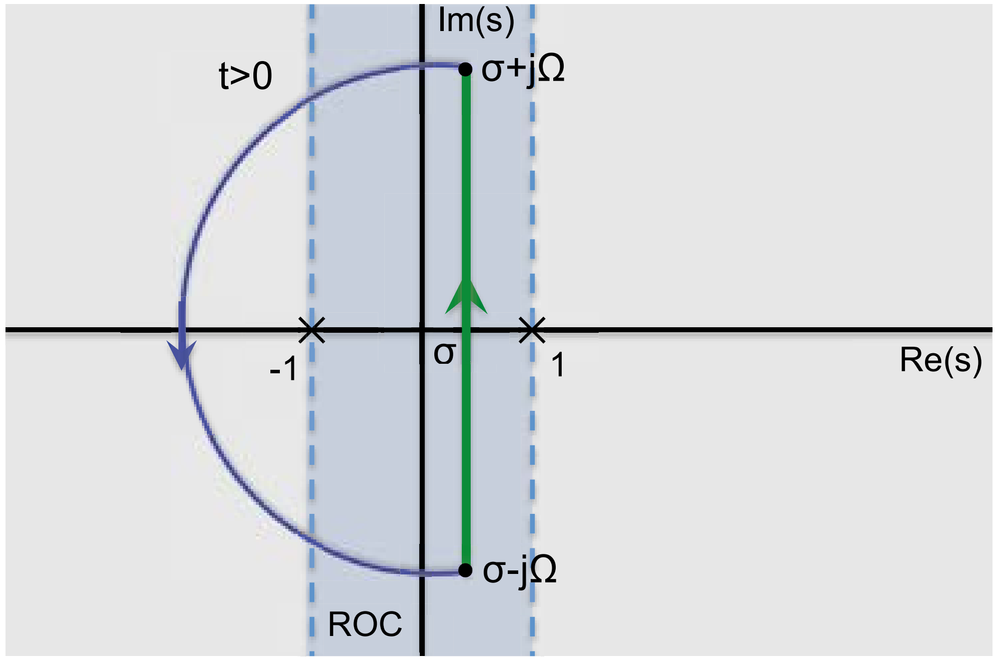
- Conclusion: $x(t) = r(t)$

- **Example 4** Suppose
$$X(s) = \frac{2}{1 - s^2}$$
with an ROC given by $\mathrm{ROC}_x = \{s \in \mathbb{C} | \, |\mathrm{Re}(s)| < 1\}$
- What is the corresponding time signal $x(t)$?
- As always, the Bromwich contour is located within the ROC

The Inverse Laplace Transform

- $X(s)$ has two simple poles: one at $s = -1$ and one at $s = +1$
- The pole at $s = 1$ is located to the right of the Bromwich contour and contributes for $t < 0$
- The pole at $s = -1$ is located to the left of the Bromwich contour and contributes for $t > 0$





The Inverse Laplace Transform

- To compute the time-domain signal for $t < 0$, we compute the residue at $s = 1$
- First, determine the φ-function

$$\varphi(s) = (s-1)X(s)e^{st} = -2\frac{e^{st}}{s+1}$$

- The residue of $X(s)e^{st}$ at $s = 1$ now follows as

$$\text{Res}\left[X(s)e^{st}, 1\right] = \left.\frac{\varphi(s)}{0!}\right|_{s=1} = -e^{t}$$

- Substitution in the residue formula for $t < 0$ gives $x(t) = e^{t}$ for $t < 0$

The Inverse Laplace Transform

- To determine the time-domain signal for $t > 0$, we compute the residue at $s = -1$
- First, the φ-function

$$\varphi(s) = (s+1)X(s)e^{st} = -2\frac{e^{st}}{s-1}$$

- The residue of $X(s)e^{st}$ at $s = -1$ is

$$\text{Res}\left[X(s)e^{st}, -1\right] = \left.\frac{\varphi(s)}{0!}\right|_{s=-1} = e^{-t}$$

- Substitution in the residue formula for $t > 0$ gives $x(t) = e^{-t}$ for $t > 0$
- Conclusion: $x(t) = e^{-|t|}$

The Inverse Laplace Transform

- To evaluate the inversion formula, we have restricted ourselves to strictly proper rational functions
- However, contour integration techniques can be applied to a much wider class of functions
- For example, suppose that the transfer function of a causal LTI system is given by

$$H(s) = \frac{1}{\sqrt{s}} \qquad \text{with} \quad \mathrm{ROC}_h = \{s \in \mathbb{C} | \mathrm{Re}(s) > 0\}$$

The Inverse Laplace Transform

- Using contour integration, it is possible to show that the corresponding impulse response is

$$h(t) = \frac{1}{\sqrt{\pi t}} u(t)$$

- We will not consider such signals in this course ($H(s)$ is not a rational function)
- As an aside: Is this LTI system BIBO stable?

The Inverse Laplace Transform

- In our analysis, we have restricted ourselves to strictly proper rational Laplace domain functions

$$H(s) = \frac{p_M(s)}{q_N(s)}$$

- $p_M(s)$ is a polynomial of degree M
- $q_N(s)$ is a polynomial of degree N
- $M < N$

The Inverse Laplace Transform

- To explain why this covers many cases of practical interest, we return to the ordinary differential equation

$$\left(a_N \frac{\mathrm{d}^N}{\mathrm{d}t^N} + a_{N-1}\frac{\mathrm{d}^{N-1}}{\mathrm{d}t^{N-1}} + \ldots + a_1 \frac{\mathrm{d}}{\mathrm{d}t} + a_0\right) y(t) = \left(b_M \frac{\mathrm{d}^M}{\mathrm{d}t^M} + b_{M-1}\frac{\mathrm{d}^{M-1}}{\mathrm{d}t^{M-1}} + \ldots + b_1 \frac{\mathrm{d}}{\mathrm{d}t} + b_0\right) x(t)$$

which holds for $t > 0^-$ and has to be supplemented by a set of initial conditions (see Lecture 2)

The Inverse Laplace Transform

- We note that the coefficients a_i and b_j are all real-valued
- For vanishing initial conditions, the solution of the above equation is called the zero-state response
- For vanishing initial conditions, the system that is described by the differential equation is LTI

The Inverse Laplace Transform

- Applying a one-sided Laplace transformation to the differential equation and taking the vanishing initial conditions into account, we find

$$\left(a_N s^N + a_{N-1} s^{N-1} + \ldots + a_1 s + a_0\right) Y(s) = \left(b_M s^M + b_{M-1} s^{N-1} + \ldots + b_1 s + b_0\right) X(s)$$

- or

$$q_N(s)\, Y(s) = p_M(s)\, X(s)$$

The Inverse Laplace Transform

- with

$$p_M(s) = b_M s^M + b_{M-1} s^{N-1} + ... + b_1 s + b_0$$

and

$$q_N(s) = a_N s^N + a_{N-1} s^{N-1} + ... + a_1 s + a_0$$

The Inverse Laplace Transform

- The transfer function of the LTI system is

$$H(s) = \frac{Y(s)}{X(s)} = \frac{p_M(s)}{q_N(s)},$$

which is a rational function in s

The Inverse Laplace Transform

- We repeat
  * For $M > N$ the transfer function is an *improper* rational function
  * For $M \leq N$ the transfer function is a *proper* rational function
  * For $M < N$ the transfer function is a *strictly proper* rational function

The Inverse Laplace Transform

- Now it can be shown that if H is proper or improper then it can always be written as

$$H(s) = R_{M-N}(s) + \frac{S(s)}{T(s)}$$

- $R_{M-N}(s)$ is a polynomial in s of degree $M - N$
- S and T are polynomials such that the rational function S/T is strictly proper

- **Example 1**

$$H(s) = \frac{s}{s+1}$$

is a proper rational function, which can be written as

$$H(s) = 1 - \frac{1}{s+1}$$

- In this example, $R_0(s) = 1$ and $-1/(s+1)$ is strictly proper

- **Example 2**

$$H(s) = \frac{s^3}{s+4}$$

is an improper rational function, which can be written as

$$H(s) = s^2 - 4s + 16 - \frac{64}{s+4}$$

- In this example, $R_2(s) = s^2 - 4s + 16$ and $-64/(s+4)$ is strictly proper

The Inverse Laplace Transform

- Time-domain signals can now be obtained using residue calculus and by identifying powers of s with derivatives in time (constants transform into Dirac distributions)
- Another approach is to expand strictly proper rational functions in partial fractions such that we can use the known transforms of standard signals to retrieve the corresponding time signals
- How to expand depends on the roots of the denominator polynomial
- We illustrate for a denominator polynomial that is quadratic

The Inverse Laplace Transform

- **Two distinct roots** (possibly complex)
- Suppose $H(s)$ is a strictly proper transfer function

$$H(s) = \frac{N(s)}{(s+p_1)(s+p_2)}, \qquad s \in \mathrm{ROC}_h, p_1 \neq p_2$$

- $N(s)$ is a polynomial of degree ≤ 1 with real coefficients

The Inverse Laplace Transform

- The partial fraction expansion of H is

$$H(s) = \frac{A_1}{s+p_1} + \frac{A_2}{s+p_2}$$

- To find A_1 and A_2, multiply $H(s)$ by $(s+p_1)(s+p_2)$. This gives

$$N(s) = A_1(s+p_2) + A_2(s+p_1)$$

- Setting $s = -p_1$, we obtain

$$A_1 = \frac{N(-p_1)}{p_2 - p_1}$$

- Setting $s = -p_2$, we obtain

$$A_2 = \frac{N(-p_2)}{p_1 - p_2}$$

The Inverse Laplace Transform

- If p_1 and p_2 are real, the time-domain signal is

$$h(t) = \left(A_1 e^{-p_1 t} + A_2 e^{-p_2 t}\right) u(t)$$

- **Example** Suppose

$$H(s) = \frac{1}{(s+1)(s+4)} = \frac{A_1}{s+1} + \frac{A_2}{s+4}$$

- Here, $N(s) = 1$, $p_1 = 1$, and $p_2 = 4$
- We find $A_1 = 1/(4-1) = 1/3$ and $A_2 = 1/(1-4) = -1/3$, and

$$H(s) = \frac{1}{3}\left(\frac{1}{s+1} - \frac{1}{s+4}\right)$$

The Inverse Laplace Transform

- The impulse response is

$$h(t) = \frac{1}{3}\left(e^{-t} - e^{-4t}\right)u(t)$$

- If p_1 and p_2 are complex, then they have to be the complex conjugate of each other, since the coefficients of the denominator polynomial are real-valued

- We write

$$p_1 = a - \mathrm{j}\Omega_0 = p_2^* \qquad a, \Omega_0 \in \mathbb{R},$$

where the asterisk denotes complex conjugation

- Recall that the coefficients of the nominator polynomial $N(s)$ are also real-valued

The Inverse Laplace Transform

- Consequently, $N^*(s) = N(s^*)$ and

$$A_2^* = \frac{N^*(-p_2)}{p_1^* - p_2^*} = \frac{N(-p_2^*)}{p_2 - p_1} = \frac{N(-p_1)}{p_2 - p_1} = A_1$$

- With $A_1 = A = A_2^*$, our partial fraction expansion becomes

$$\frac{N(s)}{(s+a)^2 + \Omega_0^2} = \frac{N(s)}{(s + \underbrace{a - \mathrm{j}\Omega_0}_{p_1})(s + \underbrace{a + \mathrm{j}\Omega_0}_{p_2})} = \frac{A}{s + a - \mathrm{j}\Omega_0} + \frac{A^*}{s + a + \mathrm{j}\Omega_0}$$

The Inverse Laplace Transform

- The corresponding time signal is

$$h(t) = Ae^{-at}e^{\mathrm{j}\Omega_0 t}u(t) + A^* e^{-at}e^{-\mathrm{j}\Omega_0 t}u(t) = 2e^{-at}\mathrm{Re}\left(Ae^{\mathrm{j}\Omega_0 t}\right)u(t)$$

- Cartesian decomposition of the complex number A:

$$A = A_\mathrm{r} + \mathrm{j}A_\mathrm{i}, \qquad A_\mathrm{r} = \mathrm{Re}(A), \quad A_\mathrm{i} = \mathrm{Im}(A)$$

- The time signal is

$$h(t) = 2e^{-at}\left[A_\mathrm{r}\cos(\Omega_0 t) - A_\mathrm{i}\sin(\Omega_0 t)\right]u(t)$$

The Inverse Laplace Transform

- Polar decomposition of the complex number A:

$$A = |A| e^{j\theta}$$

- The time signal is

$$h(t) = 2|A|\, e^{-at} \cos(\Omega_0 t + \theta)\, u(t)$$

- Both expression describe the same signal, of course

The Inverse Laplace Transform

- **Coinciding real roots**
- Suppose that the Laplace domain function is of the form

$$H(s) = \frac{N(s)}{(s+\alpha)^2}$$

- In this case, H has a double real root at $s = -\alpha$

The Inverse Laplace Transform

- Its partial fraction expansion is

$$H(s) = \frac{N(s)}{(s+\alpha)^2} = \frac{a}{(s+\alpha)^2} + \frac{b}{s+\alpha}$$

- To find a and b, we multiply by $(s+\alpha)^2$

- We obtain

$$N(s) = a + b(s + \alpha)$$

- Setting $s = -\alpha$, we find $a = N(-\alpha)$

- Substitution now gives

$$N(s) - N(-\alpha) = b(s + \alpha)$$

- Selecting a value for $s \neq -\alpha$ gives b

- For example, if $\alpha \neq 0$ we can take $s = 0$ and b follows as

$$b = \frac{N(0) - N(-\alpha)}{\alpha}$$

- The corresponding time signal is

$$h(t) = (a\,te^{-\alpha t} + be^{-\alpha t})u(t)$$

- **Example** Let

$$H(s) = \frac{4}{s(s+2)^2}$$

- Its partial fraction expansion is

$$H(s) = \frac{4}{s(s+2)^2} = \frac{A}{s} + \frac{B}{(s+2)^2} + \frac{C}{s+2}$$

- Multiplication by $s(s+2)^2$ gives

$$4 = (A+C)s^2 + (4A+B+2C)s + 4A$$

- Equating equal powers of s gives

$$A + C = 0$$
$$4A + B + 2C = 0$$
$$4A = 4$$

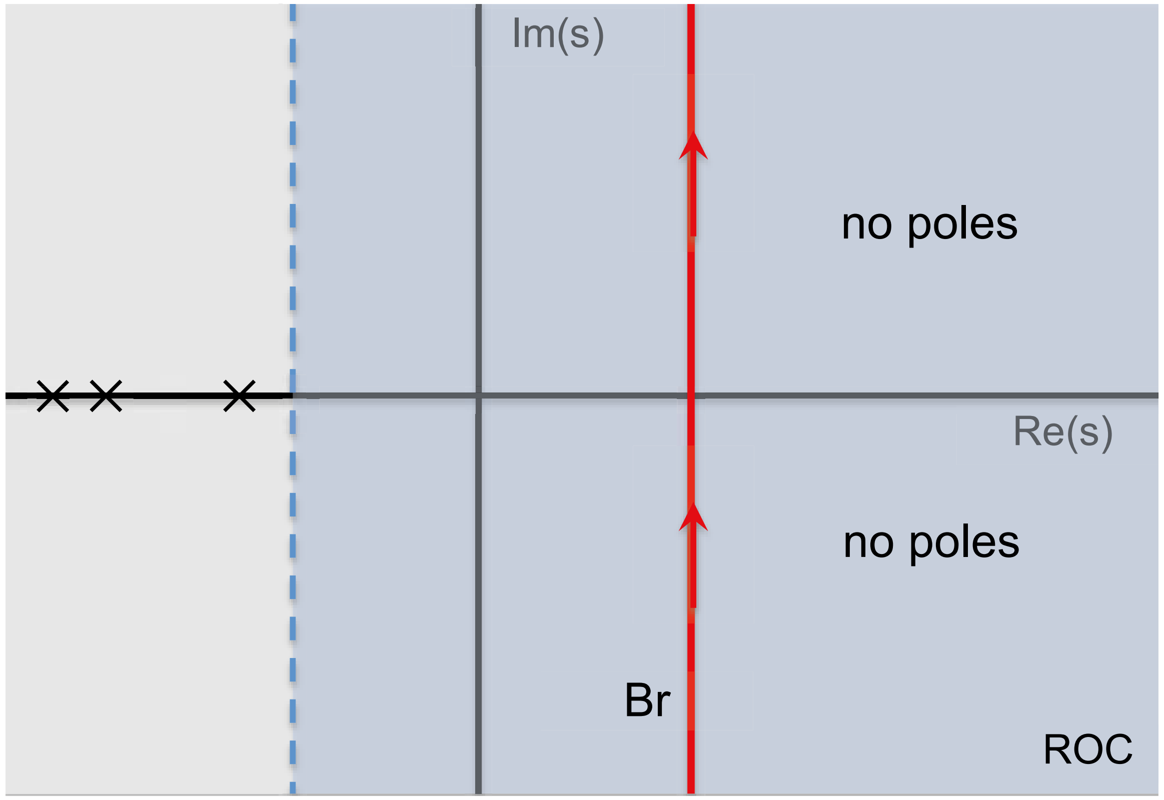
from which it follows that $A = 1$, $B = -2$, and $C = -1$

The Steady-State Response

- We are given a causal LTI system with a rational transfer function $H(s)$ and ROC_x as its region of convergence
- Also given is that the Fourier transform $H(\Omega)$ exists
- The system is then BIBO stable
- The existence of the Fourier transform implies BIBO stability for such a system

The Steady-State Response

- Let's analyze
- The ROC of a causal system is some right-half plane



The Steady-State Response

- The jΩ-axis belongs to the ROC, since $H(\Omega)$ exists
- This implies that all poles of $H(s)$ are located in the left-half of the complex s-plane
- The time signals that correspond to these poles are all exponentially decaying as time increases
- Consequently, $h(t)$ is absolutely integrable and the system is BIBO stable

The Steady-State Response

- Let $y(t)$ be the output signal of a causal LTI system due to a causal input signal
- The output signal is made up of a transient response and a steady-state response
- Transient response: signal due to the inertia of the system
- Steady-state response: signal that remains if you wait for a "sufficiently long" time (after all transients have essentially vanished)
- By studying the poles of the Laplace transform of $y(t)$, we can conclude whether or not such a steady-state response exists

The Steady-State Response

- Observations (use a Laplace transform table, if necessary):

  1. A pole in the right-half of the complex s-plane corresponds to a time signal that grows exponentially in time (irrespective of the order of the pole)
  2. A pole in the left-half of the complex s-plane corresponds to a time signal that exponentially decays to zero (irrespective of the order of the pole)
  3. A pole on the imaginary axis with an order larger than one corresponds to a time signal that shows polynomial growth in time
  4. A simple pole on the imaginary axis corresponds to a signal that remains bounded in time

The Steady-State Response

- Given these observations, we conclude that a steady-state response exists if
- $Y(s)$ has no poles in the right-half of the complex s-plane and no poles with an order larger than one on the imaginary axis
- If all poles of $Y(s)$ are in the left-half of the complex s-plane then the steady-state response vanishes

The Steady-State Response

Rigoreous proofs of the many properties of the Laplace transform (Abel's theorem, for example), the existence of the abscissa of convergence, etc. can be found in

P. Henrici, *Applied and Computational Analysis*, Vol. 2, Wiley Classics Library, New York, 1991

J. E. Marsden and M. J. Hoffman, *Basic Complex Analysis*, 2nd Ed., W. H. Freeman and Company, New York, 1987

W. R. LePage, *Complex Variables and the Laplace Transform for Engineers*, Dover Inc., New York, 1980.